

Pointwise Arbitrage Pricing Theory in Discrete Time

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Abstract

We develop a robust framework for pricing and hedging of derivative securities in discrete-time financial markets. We consider markets with both dynamically and statically traded assets and make minimal measurability assumptions. We obtain an abstract (pointwise) Fundamental Theorem of Asset Pricing and Pricing–Hedging Duality. Our results are general and in particular include so-called model independent results of Acciaio et al. (2016); Burzoni et al. (2016b) as well as seminal results of Dalang et al. (1990) in a classical probabilistic approach. Our analysis is scenario-based: a model specification is equivalent to a choice of scenarios to be considered. The choice can vary between all scenarios and the set of scenarios charged by a given probability measure. In this way, our framework interpolates between a model with universally acceptable broad assumptions and a model based on a specific probabilistic view of future asset dynamics.

1 Introduction

The *State Preference Model* or *Asset Pricing Model* underpins most mathematical descriptions of Financial Markets. It postulates that the price of d financial assets is known at a certain initial time $t_0 = 0$ (today), while the price at future times $t > 0$ is unknown and is given by a certain random outcome. To formalize such a model we only need to fix a quadruple $(X, \mathcal{F}, \mathbb{F}, S)$, where X is the set of scenarios, \mathcal{F} a σ -algebra and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in I} \subseteq \mathcal{F}$ a filtration such that the d -dimensional process $S := (S_t)_{t \in I}$ is adapted. At this stage, no probability measure is required to specify the Financial Market model $(X, \mathcal{F}, \mathbb{F}, S)$.

One of the fundamental reasons for producing such models is to assign rational prices to contracts which are not liquid enough to have a market-determined price. Rationality here is understood via the economic principle of absence of arbitrage opportunities, stating that it should not be possible to trade in the market in a way to obtain a positive gain without taking any risk. Starting from this premise, the theory of pricing by no arbitrage has been successfully developed over the last 50 years. Its cornerstone result, known as the Fundamental Theorem of Asset Pricing (FTAP), establishes equivalence between absence of arbitrage and existence of risk neutral pricing rules. The intuition for this equivalence can be accredited to de Finetti for his work on *coherence* and *previsions* (see de Finetti (1990)). The first systematic attempt to understand the absence of arbitrage opportunities in models of financial assets can be found in the works of Ross (1976, 1977)

on capital pricing, see also Huberman (1982). The intuition underpinning the arbitrage theory for derivative pricing was developed by Samuelson (1965), Black and Scholes (1973) and Merton (1973). The rigorous theory was then formalized by Harrison and Kreps (1979) and extended in Harrison and Pliska (1981), see also Kreps (1981). Their version of FTAP, in the case of a finite set of scenarios X , can be formulated as follows. Consider $X = \{\omega_1, \dots, \omega_n\}$ and let $s = (s^1, \dots, s^d)$ be the initial prices of d assets with random outcome $S(\omega) = (S^1(\omega), \dots, S^d(\omega))$ for any $\omega \in X$. Then, we have the following equivalence

$$\begin{aligned} \nexists H \in \mathbb{R}^d \text{ such that } H \cdot s \leq 0 \\ \text{and } H \cdot S(\omega) \geq 0 \text{ with } > \text{ for some } \omega \in X \end{aligned} \iff \begin{aligned} \exists Q \in \mathcal{P} \text{ such that } Q(\omega_j) > 0 \text{ and} \\ E_Q[S^i] = s^i, \forall 1 \leq j \leq n, 1 \leq i \leq d \end{aligned} \quad (1)$$

where \mathcal{P} is the class of probability measures on X . In particular, no reference probability measure is needed above and impossible events are automatically excluded from the construction of the state space X . On the other hand, linear pricing rules consistent with the observed prices s^1, \dots, s^d and the No Arbitrage condition, turn out to be (risk-neutral) probabilities with full support, that is, they assign positive measure to any state of the world. By introducing a *reference probability measure* P with full support and defining an arbitrage as a portfolio with $H \cdot s \leq 0$, $P(H \cdot S(\omega) \geq 0) = 1$ and $P(H \cdot S(\omega) > 0) > 0$, the thesis in (1) can be restated as

$$\text{There is No Arbitrage} \iff \exists Q \sim P \text{ such that } E_Q[S^i] = s^i \quad \forall i = 1, \dots, d. \quad (2)$$

The identification suggested by (2) allows non-trivial extensions of the FTAP to the case of a general space X with a fixed reference probability measure, and was proven in the celebrated work Dalang et al. (1990), by use of measurable selection arguments. It was then extended to continuous time models by Delbaen and Schachermayer (1994b,a).

The idea of introducing a reference probability measure to select scenarios proved very fruitful in the case of a general X and was instrumental for the rapid growth of the modern financial industry. It was pioneered by Samuelson (1965) and Black and Scholes (1973) who used it to formulate a continuous time financial asset model with unique rational prices for all contingent claims. Such models, with strong assumptions implying a unique derivative pricing rule, are in stark contrast to a setting with little assumptions, e.g. where the asset can follow any non-negative continuous trajectory, which are consistent with many rational pricing rules. This dichotomy was described and studied in the seminal paper of Merton (1973) who referred to the latter as “assumptions sufficiently weak to gain universal support¹” and pointed out that it typically generates outputs which are not specific enough to be of practical use. For that reason, it was the former approach, with the reference probability measure interpreted as a probabilistic description of future asset dynamics, which became the predominant paradigm in the field of quantitative finance. The original simple models were extended, driven by the need to capture additional features observed in the increasingly complex market reality, including e.g. local or stochastic volatility. Such extensions can be seen as enlarging the set of scenarios considered in the model and usually led to plurality of rational prices.

¹This setting has been often described as “model-independent” but we see it as a modelling choice with very weak assumptions.

More recently, and in particular in the wake of the financial crisis, the critique of using a single reference probability measure came from considerations of the so-called Knightian uncertainty, going back to Knight (1921), and describing *the model risk*, as contrasted with financial risks captured within a given model. The resulting stream of research aims at extending the probabilistic framework of Dalang et al. (1990) to a framework which allows for a set of possible priors $\mathcal{R} \subseteq \mathcal{P}$. The class \mathcal{R} represents a collection of plausible (probabilistic) models for the market. In continuous time models this led naturally to the theory of quasi-sure stochastic analysis as in Denis and Martini (2006); Peng (2010); Soner et al. (2011a,b). In discrete time, a general approach was developed by Bouchard and Nutz (2015). Under some technical conditions on the state space and the set \mathcal{R} they provide a version of the FTAP, as well as the superhedging duality. Their framework includes the two extreme cases: the classical case when $\mathcal{R} = \{P\}$ is a singleton and, on the other extreme, the case of full ambiguity when \mathcal{R} coincides with the whole set of probability measures and the description of the model becomes pathwise. Their setup has been used to study a series of related problems, see e.g. Bayraktar and Zhang (2016).

Describing models by specifying a family of probability measures \mathcal{R} appears natural when starting from the dominant paradigm where a reference measure P is fixed. However, it is not the only way, and possibly not the simplest one, to specify a model. Indeed, in this paper, we develop a different approach inspired by the original finite state space model used in Harrison and Pliska (1981) as well as the notion of *prediction set* in Mykland (2003), see also Hou and Obłój (2015). Our analysis is scenario based. More specifically, *agent's beliefs* or *a model* are equivalent to selecting a set of admissible scenarios which we denote by $\Omega \subseteq X$. The selection may be formulated e.g. in terms of behaviour of some market observable quantities and may reflect both the information the agent has as well as the modelling assumptions she is prepared to make. Our approach clearly includes the “universally acceptable” case of considering all scenarios $\Omega = X$ but we also show that it subsumes the probabilistic framework of Dalang et al. (1990). Importantly, as we work under minimal measurability requirement on Ω , our models offer a flexible way to interpolate between the two settings. The scenario based specification of a model requires less sophistication than selection of a family of probability measures and appears particularly natural when considering (super)-hedging which is a pathwise property.

Our first main result, Theorem 2.3, establishes a Fundamental Theorem of Asset Pricing for an arbitrary specification of a model Ω and gives equivalence between existence of a rational pricing rule (i.e. a calibrated martingale measure) and absence of, suitably defined, arbitrage opportunities. Interestingly the equivalence in (1) does not simply extend to a general setting: specification of Ω which is inconsistent with any rational pricing rule does not imply existence of *one arbitrage strategy*. Ex post, this is intuitive: while all agents may agree that rational pricing is impossible they may well disagree on why this is so. This discrepancy was first observed, and illustrated with an example, by Davis and Hobson (2007). The equivalence is only recovered under strong assumptions, as shown by Riedel (2015) in a topological one-period setup and by Acciaio et al. (2016) in a general discrete time setup. A rigorous analysis of this phenomenon in the case $\Omega = X$ was subsequently given by Burzoni et al. (2016b), who also showed that several notions of arbi-

trage can be studied within the same framework. Here, we extend their result to an arbitrary $\Omega \subset X$ and to the setting with both dynamically traded assets and statically traded assets. We show that also in such cases agents' different views on arbitrage opportunities may be aggregated in a canonical way into a pointwise arbitrage strategy in an enlarged filtration. As special cases of our general FTAP, we recover results in Acciaio et al. (2016); Burzoni et al. (2016b) as well as the classical Dalang-Morton-Willinger theorem Dalang et al. (1990). For the latter, we show that choosing a probability measure P on X is equivalent to fixing a suitable set of scenarios Ω^P and our results then lead to probabilistic notions of arbitrage as well as the probabilistic version of the Fundamental Theorem of Asset pricing.

Our second main result, Theorem 2.5, characterises the range of rational prices for a contingent claim. Our setting is comprehensive: we make no regularity assumptions on the model specification Ω , on the payoffs of traded assets, both dynamic and static, or the derivative which we want to price. We establish a pricing–hedging duality result asserting that the infimum of prices of super-hedging strategies is equal to the supremum of rational prices. As already observed in Burzoni et al. (2016a), but also in Beiglböck et al. (2016) in the context of martingale optimal transport, it may be necessary to consider superhedging on a smaller set of scenarios than Ω in order to avoid a duality gap between rational prices and superhedging prices. In this paper this feature is achieved through the set of *efficient* trajectories Ω_Φ^* which only depends on Ω and the market. The set Ω_Φ^* recollects all scenarios which are supported by some rational pricing rule. Its intrinsic and constructive characterisation is given in the FTAP, Theorem 2.3. Our duality generalizes the results of Burzoni et al. (2016a) to the setting of abstract model specification Ω as well as generic finite set of statically traded assets. The flexibility of model choice is of particular importance, as stressed above. The “universally acceptable” setting $\Omega = X$ will typically produce wide range of rational prices which may not be of practical relevance, as already discussed by Merton (1973). However, as we shrink Ω from X to a set Ω^P , the range of rational prices shrinks accordingly and, in case Ω^P corresponds to a complete market model, the interval reduces to a single point. This may be seen as a quantification of the impact of modelling assumptions on rational prices and gives a powerful description of model risk.

We note that pricing–hedging duality results have a long history in the field of robust pricing and hedging. First contributions focused on obtaining explicit results working in a setting with one dynamically traded risky asset and a strip of statically traded co-maturing call options with all strikes. In his pioneering work Hobson (1998) devised a methodology based on Skorkohod embedding techniques and treated the case of lookback options. His approach was then used in a series of works focusing on different classes of exotic options, see Brown et al. (2001); Cox and Oblój (2011b,a); Cox and Wang (2013); Hobson and Klimmek (2013); Hobson and Neuberger (2012); Henry-Labordère et al. (2016). More recently, it has been re-cast as an optimal transportation problem along martingale dynamics and the focus shifted to establishing abstract pricing–hedging duality, see Beiglböck et al. (2013); Davis et al. (2014); Dolinsky and Soner (2014); Hou and Oblój (2015).

The remainder of the paper is organised as follows. First, in Section 2, we present all the main results. We give the necessary definitions and in Section 2.1 state our two main theorems: the Fundamental Theorem of Asset Pricing, Theorem 2.3, and the pricing–hedging duality, Theorem 2.5, which we also refer to as the superhedging duality. In Section 2.2, we generalize the results of Acciaio et al. (2016) for a multi-dimensional non-canonical stock process. Here, suitable continuity assumptions and presence of a statically traded option ϕ_0 with convex payoff with superlinear growth allow to “lift” superhedging from Ω_Φ^* to the whole Ω . Finally, in Section 2.3, we recover the classical probabilistic results of Dalang et al. (1990). The rest of the paper then discusses the methodology and the proofs. Section 3 is devoted to the construction of strategy and filtration which aggregate arbitrage opportunities seen by different agents. We first treat the case without statically traded options when the so-called Arbitrage Aggregator is obtained through a conditional backwards induction. Then, when statically traded options are present, we devise a Pathspace Partition Scheme, which iteratively identifies the class of polar sets with respect to calibrated martingale measure. Section 4 contains the proofs with some technical remarks relegated to the Appendix.

2 Main Results

We work on a Polish space X and denote \mathcal{B}_X its Borel sigma-algebra and \mathcal{P} the set of all probability measures on (X, \mathcal{B}_X) . If $\mathcal{G} \subseteq \mathcal{B}_X$ is a sigma algebra and $P \in \mathcal{P}$, we denote with $\mathcal{N}^P(\mathcal{G}) := \{N \subseteq A \in \mathcal{G} \mid P(A) = 0\}$ the class of P -null sets from \mathcal{G} . We denote \mathcal{F}^A the sigma algebra generated by the analytic sets of (X, \mathcal{B}_X) (see Dellacherie and Meyer (1982) Chapter III). Any $P \in \mathcal{P}$ extends in a unique way to a measure on \mathcal{F}^A which, with a slight abuse of notation, we still denote P . As discussed in the introduction, we consider pointwise arguments and think of a model as a choice of universe of scenarios $\Omega \subset X$. Throughout, we assume that $\Omega \in \mathcal{F}^A$.

Given a family of measures $\mathcal{R} \subseteq \mathcal{P}$ we say that a set is polar (with respect to \mathcal{R}) if it belongs to $\{N \subseteq A \in \mathcal{B}_X \mid Q(A) = 0 \ \forall Q \in \mathcal{R}\}$ and a property is said to hold *quasi surely* (\mathcal{R} -q.s.) if it holds outside a polar set. For those random variables g whose positive and negative part is not Q -integrable ($Q \in \mathcal{P}$) we adopt the convention $\infty - \infty = -\infty$ when we write $E_Q[g] = E_Q[g^+] - E_Q[g^-]$. Finally for any sigma-algebra \mathcal{G} we shall denote by $\mathcal{L}(X, \mathcal{G}; \mathbb{R}^d)$ the space of \mathcal{G} -measurable d -dimensional random vectors. For a given set $A \in \mathcal{F}^A$ and $f, g \in \mathcal{L}(X, \mathcal{G}; \mathbb{R})$ we will often refer to $X \leq Y$ on A whenever $f(\omega) \leq g(\omega)$ for every $\omega \in A$ (similarly for $=$ and $<$).

We fix a time horizon $T \in \mathbb{N}$ and let $\mathbb{T} := \{0, 1, \dots, T\}$. We assume the market includes both liquid assets, which can be traded dynamically through time, and less liquid assets which are only available for trading at time $t = 0$. The prices of assets are represented by an \mathbb{R}^d -valued stochastic process $S = (S_t)_{t \in \mathbb{T}}$ on (X, \mathcal{B}_X) . In addition we may also consider the existence of a vector of non-traded assets represented by an \mathbb{R}^{d_n} -valued stochastic process $Y = (Y_t)_{t \in \mathbb{T}}$ on (X, \mathcal{B}_X) , which can also be interpreted as market factors, or additional information available to the agent. The prices are given in units of some fixed numeraire asset S^0 , which itself is thus normalized: $S_t^0 = 1$ for all $t \in \mathbb{T}$. In the presence of the additional factor Y , we let $\mathbb{F}^{S,Y} := (\mathcal{F}_t^{S,Y})_{t \in \mathbb{T}}$ be the natural filtration generated by S and Y (Similarly $\mathbb{F}^S := (\mathcal{F}_t^S)_{t \in \mathbb{T}}$ if Y is not considered). For technical reasons, we

will also make use of the filtration $\mathbb{F}^{\mathcal{A}} := (\mathcal{F}_t^{\mathcal{A}})_{t \in \mathbb{T}}$ where $\mathcal{F}_t^{\mathcal{A}}$ is the sigma algebra generated by the analytic sets of $(X, \mathcal{F}_t^{S,Y})$ (see Dellacherie and Meyer (1982) Chapter III). Clearly, $\mathbb{F}^{S,Y} \subset \mathbb{F}^{\mathcal{A}}$. Finally, we let Φ denote the vector of payoffs of the statically traded assets. We consider the setting when $\Phi = \{\phi_1, \dots, \phi_k\}$ is finite and each $\phi \in \Phi$ is $\mathcal{F}^{\mathcal{A}}$ -measurable. When there are no statically traded assets we set $\Phi = 0$ which makes our notation consistent.

For any filtration \mathbb{F} , $\mathcal{H}(\mathbb{F})$ is the class of \mathbb{F} -predictable stochastic processes, with values in \mathbb{R}^d , which represent admissible trading strategies. Gains from investing in S , adopting a strategy H , are given by $(H \circ S)_T := \sum_{t=1}^T \sum_{j=1}^d H_t^j (S_t^j - S_{t-1}^j) = \sum_{t=1}^T H_t \cdot \Delta S_t$. In contrast, ϕ_j can only be bought or sold at time $t = 0$ (without loss of generality with zero initial cost) and held until the maturity T , so that trading strategies are given by $h \in \mathbb{R}^k$ and generate payoff $h \cdot \Phi := \sum_{j=1}^k h_j \phi_j$ at time T . We let $\mathcal{A}_{\Phi}(\mathbb{F})$ denote the set of such \mathbb{F} -admissible trading strategies (h, H) .

Given a filtration \mathbb{F} , universe of scenarios Ω and set of statically traded assets Φ , we let

$$\mathcal{M}_{\Omega, \Phi}(\mathbb{F}) := \{Q \in \mathcal{P} \mid S \text{ is an } \mathbb{F}\text{-martingale under } Q, Q(\Omega) = 1 \text{ and } E_Q[\phi] = 0 \forall \phi \in \Phi\}.$$

The support of a probability measure Q is given by $\text{supp}(Q) := \bigcap \{C \in \mathcal{F} \mid C \text{ closed, } P(C) = 1\}$. We often consider measures with finite support and denote it with a superscript f , i.e. for a given set \mathcal{R} of probability measures we put $\mathcal{R}^f := \{Q \in \mathcal{R} \mid \text{supp}(Q) \text{ is finite}\}$. To wit, $\mathcal{M}_{\Omega, \Phi}^f(\mathbb{F})$ denotes finitely supported martingale measures on Ω which are calibrated to options in Φ . Define

$$\mathbb{F}^M := (\mathcal{F}_t^M)_{t \in \mathbb{T}}, \quad \text{where } \mathcal{F}_t^M := \bigcap_{P \in \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^{S,Y})} \mathcal{F}_t^{S,Y} \vee \mathcal{N}^P(\mathcal{F}_T^{S,Y}). \quad (3)$$

Remark 2.1 *In this paper we only consider filtrations \mathbb{F} which satisfy $\mathbb{F}^{S,Y} \subseteq \mathbb{F} \subseteq \mathbb{F}^M$. All such filtrations generate the same set of martingale measures, in the sense that any $Q \in \mathcal{M}_{\Omega, \Phi}(\mathbb{F})$ uniquely extends to a measure $\hat{Q} \in \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^M)$ and, reciprocally, for any $\hat{Q} \in \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^M)$, the restriction $\hat{Q}|_{\mathbb{F}}$ belongs to $\mathcal{M}_{\Omega, \Phi}(\mathbb{F})$. Accordingly, with a slight abuse of notation, we will write $\mathcal{M}_{\Omega, \Phi}(\mathbb{F}) = \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^M) = \mathcal{M}_{\Omega, \Phi}$.*

In the subsequent analysis, the set of scenarios charged by martingale measures is crucial:

$$\Omega_{\Phi}^* := \left\{ \omega \in \Omega \mid \exists Q \in \mathcal{M}_{\Omega, \Phi}^f \text{ such that } Q(\omega) > 0 \right\} = \bigcup_{Q \in \mathcal{M}_{\Omega, \Phi}^f} \text{supp}(Q). \quad (4)$$

We have by definition that for every $Q \in \mathcal{M}_{\Omega, \Phi}^f$ its support satisfies $\text{supp}(Q) \subseteq \Omega_{\Phi}^*$. Notice that the key elements introduced so far namely $\mathcal{M}_{\Omega, \Phi}$, $\mathcal{M}_{\Omega, \Phi}^f$, \mathbb{F}^M and Ω_{Φ}^* , only depend on the four basic ingredients of the market: Ω , S , Y and Φ . Finally, in all of the above notations, we omit the subscript Ω when $\Omega = X$ and we omit the subscript Φ when $\Phi = 0$, e.g. \mathcal{M}^f denotes all finitely supported martingale measures on X .

2.1 Fundamental Theorem of Asset Pricing and Superhedging Duality

We now introduce different notions of arbitrage opportunities which play a key role in the statement of the pointwise Fundamental Theorem of Asset Pricing.

Definition 2.2 Fix a filtration \mathbb{F} , $\Omega \in \mathcal{F}^{\mathcal{A}}$ and a set of statically traded options Φ .

1p A One-Point Arbitrage (1p-Arbitrage) is a strategy $(\alpha, H) \in \mathcal{A}_{\Phi}(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T \geq 0$ on Ω with a strict inequality for some $\omega \in \Omega$.

SA A Strong Arbitrage is a strategy $(\alpha, H) \in \mathcal{A}_{\Phi}(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T > 0$ on Ω .

USA A Uniformly Strong Arbitrage is a strategy $(\alpha, H) \in \mathcal{A}_{\Phi}(\mathbb{F})$ such that $\alpha \cdot \Phi + (H \circ S)_T > \varepsilon$ on Ω , for some $\varepsilon > 0$.

Clearly, the above notions are relative to the inputs and we often stress this and refer to an arbitrage *in* $\mathcal{A}_{\Phi}(\mathbb{F})$ and *on* Ω . We are now ready to state the pathwise version of Fundamental Theorem of Asset Pricing. It generalizes Theorem 1.3 in Burzoni et al. (2016b) in two directions: we include an arbitrary selection of scenarios $\Omega \in \mathcal{F}^{\mathcal{A}}$ and we include static trading in options as well as dynamic trading in S .

Theorem 2.3 (Pointwise FTAP on $\Omega \subseteq X$) Fix $\Omega \in \mathcal{F}^{\mathcal{A}}$ and Φ a finite set of statically traded options. Then, there exists a filtration $\tilde{\mathbb{F}}$ which aggregates arbitrage views in that:

$$\text{No Strong Arbitrage in } \mathcal{A}_{\Phi}(\tilde{\mathbb{F}}) \text{ on } \Omega \iff \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^{S, Y}) \neq \emptyset \iff \Omega_{\Phi}^* \neq \emptyset$$

and $\mathbb{F}^{S, Y} \subseteq \tilde{\mathbb{F}} \subseteq \mathbb{F}^M$. Further, $\Omega_{\Phi}^* \in \mathcal{F}^{\mathcal{A}}$ and there exists a trading strategy $(\alpha^*, H^*) \in \mathcal{A}_{\Phi}(\tilde{\mathbb{F}})$ which is an Arbitrage Aggregator in that $\alpha^* \cdot \Phi + (H^* \circ S)_T \geq 0$ on Ω and

$$\Omega_{\Phi}^* = \{\omega \in \Omega \mid \alpha^* \cdot \Phi(\omega) + (H^* \circ S)_T(\omega) = 0\}. \quad (5)$$

Moreover, one may take $\tilde{\mathbb{F}}$ and (α^*, H^*) as constructed in (20) and (19) respectively.

Remark 2.4 We stress that $\mathcal{M}_{\Omega, \Phi} = \emptyset$ does not imply existence of a Strong Arbitrage in $\mathcal{A}_{\Phi}(\mathbb{F}^{S, Y})$ – this is true only under additional strong assumptions, see Theorem 2.9 and Example 3.13 below. In general, the former corresponds to a situation when all agents agree that rational option pricing is not possible but they may disagree on why this is so. Our result shows that any such arbitrage views can be aggregated into one strategy (α^*, H^*) in a filtration $\tilde{\mathbb{F}}$ which does not perturb the calibrated martingale measures, see Remark 2.1 above.

In fact, the proof of Theorem 2.3 relies on an explicit — up to a measurable selection — construction of the enlarged filtration $\tilde{\mathbb{F}}$ and the Arbitrage Aggregator strategy (α^*, H^*) . Even in the case when $\Phi = 0$, the proof is novel and simpler than the arguments in Burzoni et al. (2016b). In particular, it allows to conclude that H_t^* is $\mathcal{F}^{\mathcal{A}}$ -measurable for any $t = 1, \dots, T$, see Lemmas 3.5 and 3.1.

We turn now to our second main result. For a given set of scenarios $A \in \mathcal{F}^{\mathcal{A}}$, define the superhedging price on A :

$$\pi_{A, \Phi}(g) := \inf \{x \in \mathbb{R} \mid \exists (\alpha, H) \in \mathcal{A}_{\Phi}(\mathbb{F}^{\mathcal{A}}) \text{ such that } x + \alpha \cdot \Phi + (H \circ S)_T \geq g \text{ on } A\}. \quad (6)$$

Following the intuition in Burzoni et al. (2016a), we expect to obtain pricing–hedging duality only when considering superhedging on the set of scenarios visited by martingales, i.e. we consider $\pi_{\Omega_{\Phi}^*, \Phi}(g)$.

Theorem 2.5 Fix $\Omega \in \mathcal{F}^A$ and Φ a finite set of statically traded options. Then, for any \mathcal{F}^A -measurable g

$$\pi_{\Omega_\Phi^*, \Phi}(g) = \sup_{Q \in \mathcal{M}_{\Omega, \Phi}^f} E_Q[g] \quad (7)$$

and, if finite, the left hand side is attained by some strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^A)$.

The proofs of the two main theorems are given in Section 4 below. We first prove the results when $\Phi = 0$ and then extend iterating on number of statically traded options k . The proofs are intertwined and we explain their logic at the beginning of Section 4. In particular, the following proposition, which characterizes when the set of scenarios visited by martingale measures Ω_Φ^* coincides with the whole set Ω , is important.

Proposition 2.6 Fix $\Omega \in \mathcal{F}^A$. Then there are no One-Point Arbitrages on Ω with respect to \mathbb{F}^A if and only if $\Omega = \Omega_\Phi^*$.

Under a mild assumption, this situation has further equivalent characterisations:

Remark 2.7 Under the additional assumption that Φ is not perfectly replicable on Ω , the following are easily shown to be equivalent:

- (1) No One-Point Arbitrage on Ω with respect to \mathbb{F}^A .
- (2) For any $x \in \mathbb{R}^k$, when $\varepsilon_x > 0$ is small enough, $\mathcal{M}_{\Omega, \Phi + \varepsilon_x x}^f \neq \emptyset$.
- (3) When $\varepsilon > 0$ is small enough, for any $x \in \mathbb{R}^k$ such that $|x| < \varepsilon$, $\mathcal{M}_{\Omega, \Phi + x}^f \neq \emptyset$.

where $\Phi + x = \{\phi_1 + x_1, \dots, \phi_n + x_n\}$.

In particular, small uniform modifications of the statically traded options do not affect the existence of calibrated martingale measures.

2.2 Pointwise FTAP for arbitrary many options in the spirit of Acciaio et al. (2016)

In this section, we want to recover and extend the main results in Acciaio et al. (2016). A similar result can be also found in Cheridito et al. (2016) under slightly different assumptions. We work in the same setup as above except that we can allow for a larger, possibly uncountable, set of statically traded options $\Phi = \{\phi_i : i \in I\}$. Trading strategies $(h, H) \in \mathcal{A}_\Phi(\mathbb{F}^A)$ correspond to dynamic trading in S using $H \in \mathcal{H}(\mathbb{F})$ combined with a static position in a finite number of options in Φ .

Assumption 2.8 In this section, we assume that S takes values in $\mathbb{R}_+^{d \times (T+1)}$ and all the options $\phi \in \Phi$ are continuous derivatives on the underlying assets S , more precisely

$$\phi_i = g_i \circ S \quad \text{for some continuous } g_i : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}, \quad \forall i \in I.$$

In addition, we assume $0 \in I$ and $\phi_0 = g_0(S_T)$ for a strictly convex super-linear function g_0 on \mathbb{R}^d , such that other options have a slower growth at infinity:

$$\lim_{|x| \rightarrow \infty} \frac{g_i(x)}{m(x)} = 0, \quad \forall i \in I/\{0\}, \quad \text{where } m(x_0, \dots, x_T) := \sum_{t=0}^T g_0(x_t).$$

The option ϕ_0 **can be only bought** at time $t = 0$. Therefore admissible trading strategies $\mathcal{A}_\Phi(\mathbb{F})$ consider only positive values for the static position in ϕ_0 .

The presence of ϕ_0 has the effect of restricting non-trivial considerations to a compact set of values for S and then the continuity of g_i allows to aggregate different arbitrages without enlarging the filtration. This results in the following special case of the pathwise Fundamental Theorem of Asset Pricing. Denote by $\widetilde{\mathcal{M}}_{\Omega, \Phi} := \{Q \in \mathcal{M}_{\Omega, \Phi \setminus \{\phi_0\}} \mid E_Q[\phi_0] \leq 0\}$.

Theorem 2.9 *Consider $\Omega \in \mathcal{F}^A$ such that $\Omega = \Omega^*$, $\pi_{\Omega^*}(\phi_0) > 0$ and there exists $\omega^* \in \Omega$ such that $S_0(\omega^*) = S_1(\omega^*) = \dots = S_T(\omega^*)$. Under Assumption 2.8, the following are equivalent:*

- (1) *There is no Uniformly Strong Arbitrage on Ω in $\mathcal{A}_\Phi(\mathbb{F}^A)$;*
- (2) *There is no Strong Arbitrage on Ω in $\mathcal{A}_\Phi(\mathbb{F}^A)$;*
- (3) $\widetilde{\mathcal{M}}_{\Omega, \Phi} \neq \emptyset$.

Moreover, when any of these holds, for any upper semi-continuous $g : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}$ that satisfies

$$\lim_{|x| \rightarrow \infty} \frac{g^+(x)}{m(x)} = 0, \quad (8)$$

the following pricing–hedging duality holds:

$$\pi_{\Omega, \Phi}(g(S)) = \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[g(S)]. \quad (9)$$

Remark 2.10 *We show in Remark 3.14 below that the pricing–hedging duality may fail in general when super-replicating on the whole set Ω as in (9). This confirms the intuition that the the existence of an option ϕ_0 which satisfies the hypothesis of Theorem 2.9 is crucial.*

2.3 Classical Model Specific setting and its selection of scenarios

In this section we are interested in the relation of our results with the classical Dalang, Morton and Willinger approach from Dalang et al. (1990). For simplicity, and ease of comparison, throughout this section we restrict to dynamic trading only: $\Phi = 0$. We let $\mathbb{F}^\mathbb{P}$ be the \mathbb{P} -completion of \mathbb{F} . Recall that a (classical) \mathbb{P} -arbitrage is a strategy $H \in \mathcal{A}(\mathbb{F}^\mathbb{P})$ such that $(H \circ S)_T \geq 0$ \mathbb{P} -a.s. and $\mathbb{P}((H \circ S)_T > 0) > 0$.

Proposition 2.11 *Consider a probability measure $\mathbb{P} \in \mathcal{P}$ and let $\mathcal{M}^{\ll \mathbb{P}} := \{Q \in \mathcal{M} \mid Q \ll \mathbb{P}\}$. There exists a set of scenarios $\Omega^\mathbb{P} \in \mathcal{F}^A$ and a filtration $\widetilde{\mathbb{F}}$ such that $\mathbb{F}^{S, Y} \subseteq \widetilde{\mathbb{F}} \subseteq \mathbb{F}^M$ and*

$$\text{No Strong Arbitrage in } \mathcal{A}(\widetilde{\mathbb{F}}) \text{ on } \Omega^\mathbb{P} \iff \mathcal{M}^{\ll \mathbb{P}} \neq \emptyset.$$

Further,

$$\text{No } \mathbb{P}\text{-arbitrage} \iff \mathbb{P}((\Omega^\mathbb{P})^*) = 1 \iff \mathcal{M}^{\sim \mathbb{P}} \neq \emptyset,$$

where $\mathcal{M}^{\sim \mathbb{P}} := \{Q \in \mathcal{M} \mid Q \sim \mathbb{P}\}$.

Proof. For $1 \leq t \leq T$, we denote χ_{t-1} the random set $\chi_{\mathcal{G}}$ from (38) with $\xi = \Delta S_t$ and $\mathcal{G} = \mathcal{F}_{t-1}^{S,Y}$ (see Appendix 5.1 for further details). Consider now the set

$$U := \bigcap_{t=1}^T \{\omega \in X \mid \Delta S_t(\omega) \in \chi_{t-1}(\omega)\}$$

and note that, by Lemma 5.2, $U \in \mathcal{B}_X$ and $\mathbb{P}(U) = 1$. Consider now the set U^* defined as in (4) (using U in the place of Ω and for $\Phi = 0$) and define

$$\Omega^{\mathbb{P}} := \begin{cases} U & \text{if } \mathbb{P}(U^*) > 0 \\ U \setminus U^* & \text{if } \mathbb{P}(U^*) = 0 \end{cases},$$

which satisfies $\Omega^{\mathbb{P}} \in \mathcal{F}^A$ and $\mathbb{P}(\Omega^{\mathbb{P}}) = 1$.

In both the proofs of sufficiency and necessity the existence of the technical filtration is consequence of Theorem 2.3. To prove sufficiency let $Q \in \mathcal{M}^{\ll \mathbb{P}}$ and observe that, since $Q(\Omega^{\mathbb{P}}) = 1$, we have $\mathcal{M}_{\Omega^{\mathbb{P}}} \neq \emptyset$. From Theorem 2.3 we have No $(\Omega^{\mathbb{P}}, \tilde{\mathbb{P}})$ Strong Arbitrage. To prove necessity observe first that $(\Omega^{\mathbb{P}})^*$ is either equal to U^* , if $\mathbb{P}(U^*) > 0$, or to the empty set otherwise. From Theorem 2.3, No $(\Omega^{\mathbb{P}}, \tilde{\mathbb{P}})$ Strong Arbitrage implies $(\Omega^{\mathbb{P}})^* \neq \emptyset$ and thus $\mathbb{P}((\Omega^{\mathbb{P}})^*) = \mathbb{P}(U^*) > 0$. Note now that by considering $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot \mid (\Omega^{\mathbb{P}})^*)$ we have, by construction $0 \in ri(\chi_{t-1}) \tilde{\mathbb{P}} - a.s.$ for every $1 \leq t \leq T$, where $ri(\cdot)$ denotes the relative interior of a set. By Rokhlin Rokhlin (2008) we conclude that $\tilde{\mathbb{P}}$ admits an equivalent martingale measure and hence the thesis. The last statement then also follows. ■

3 Construction of the Arbitrage Aggregator and its filtration

3.1 The case without statically traded options

The following Lemma is an empowered version of Lemma 4.4 in Burzoni et al. (2016b), which relies on measurable selections arguments, instead of a pathwise explicit construction. In the following we will set $\Delta S_t = S_t - S_{t-1}$ and Σ_{t-1}^{ω} the level set of the trajectory ω up to time $t-1$ of both traded and non-traded assets, i.e.

$$\Sigma_{t-1}^{\omega} = \{\tilde{\omega} \in X \mid S_{0:t-1}(\tilde{\omega}) = S_{0:t-1}(\omega) \text{ and } Y_{0:t-1}(\tilde{\omega}) = Y_{0:t-1}(\omega)\}, \quad (10)$$

where $S_{0:t-1} := (S_0, \dots, S_{t-1})$ and $Y_{0:t-1} := (Y_0, \dots, Y_{t-1})$.

Lemma 3.1 *Fix any $t \in \{1, \dots, T\}$ and $\Gamma \in \mathcal{F}^A$. There exist an index $\beta \in \{0, \dots, d\}$, random vectors $H^1, \dots, H^{\beta} \in \mathcal{L}(X, \mathcal{F}_{t-1}^A; \mathbb{R}^d)$, \mathcal{F}_t^A -measurable sets E^0, \dots, E^{β} such that the sets $B^i := E^i \cap \Gamma$, $i = 0, \dots, \beta$, form a partition of Γ satisfying:*

1. *if $\beta > 0$ and $i = 1, \dots, \beta$ then: $B^i \neq \emptyset$; $H^i \cdot \Delta S_t(\omega) > 0$ for all $\omega \in B^i$ and $H^i \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in \bigcup_{j=i}^{\beta} B^j \cup B^0$.*
2. *$\forall H \in \mathcal{L}(X, \mathcal{F}_{t-1}^A; \mathbb{R}^d)$ such that $H \cdot \Delta S_t \geq 0$ on B^0 we have $H \cdot \Delta S_t = 0$ on B^0 .*

Remark 3.2 Clearly if $\beta = 0$ then $B^0 = \Gamma$ (which include the trivial case $\Gamma = \emptyset$). Notice also that for any $\Gamma \in \mathcal{F}^A$ and $t = \{1, \dots, T\}$ we have that $H^i = H_t^{i,\Gamma}$, $B^i = B_t^{i,\Gamma}$, $\beta = \beta_t^\Gamma$ depend explicitly on t and Γ .

Proof. Fix $t \in \{1, \dots, T\}$ and consider, for an arbitrary $\Gamma \in \mathcal{F}^A$ the multifunction

$$\psi_{t,\Gamma} : \omega \in X \mapsto \{\Delta S_t(\tilde{\omega}) \mathbf{1}_\Gamma(\tilde{\omega}) \mid \tilde{\omega} \in \Sigma_{t-1}^\omega\} \subseteq \mathbb{R}^d$$

where Σ_{t-1}^ω is defined in (10). We first show that $\psi_{t,\Gamma}$ is an \mathcal{F}_{t-1}^A -measurable multifunction. For any open set $O \subseteq \mathbb{R}^d$

$$\{\omega \in X \mid \psi_{t,\Gamma}(\omega) \cap O \neq \emptyset\} = (S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)),$$

where $B = (\Delta S_t \mathbf{1}_\Gamma)^{-1}(O)$. First $\Delta S_t \mathbf{1}_\Gamma$ is an \mathcal{F}^A -measurable random vector then $B \in \mathcal{F}^A$. Second S_u, Y_u are a Borel measurable functions for any $0 \leq u \leq t-1$ so that, from Lemma 5.1, we have that $(S_{0:t-1}, Y_{0:t-1})(B)$ belongs to the sigma-algebra generated by the analytic sets in $Mat((d+d_n) \times t; \mathbb{R})$ endowed with its Borel sigma-algebra. Applying again Lemma 5.1 we deduce that $(S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)) \in \mathcal{F}_{t-1}^A$ and hence the desired measurability for $\psi_{t,\Gamma}$. Let \mathbb{S}^d be the unit sphere in \mathbb{R}^d , by preservation of measurability (see Rockafellar and Wets (1998), Chapter 14-B) the following multifunction is closed valued and \mathcal{F}_{t-1}^A -measurable

$$\psi_{t,\Gamma}^*(\omega) := \{H \in \mathbb{S}^d \mid H \cdot y \geq 0 \quad \forall y \in \psi_{t,\Gamma}(\omega)\}.$$

It follows that it admits a Castaing representation (see Theorem 14.5 in Rockafellar and Wets (1998)), that is, there exists a countable collection of measurable functions $\{\xi_{t,\Gamma}^n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}(X, \mathcal{F}_{t-1}^A; \mathbb{R}^d)$ such that $\psi_{t,\Gamma}^*(\omega) = \overline{\{\xi_{t,\Gamma}^n(\omega) \mid n \in \mathbb{N}\}}$ for every ω such that $\psi_{t,\Gamma}^*(\omega) \neq \emptyset$ and $\xi_{t,\Gamma}^n(\omega) = 0$ for every ω such that $\psi_{t,\Gamma}^*(\omega) = \emptyset$. Recall that every $\xi_{t,\Gamma}^n$ is a measurable selector of $\psi_{t,\Gamma}^*$ and hence, $\xi_{t,\Gamma}^n \cdot \Delta S_t \geq 0$ on Γ . Note moreover that,

$$\forall \omega \in X, \quad \bigcup_{\xi \in \psi_{t,\Gamma}^*(\omega)} \{y \in \mathbb{R}^d \mid \xi \cdot y > 0\} = \bigcup_{n \in \mathbb{N}} \{y \in \mathbb{R}^d \mid \xi_{t,\Gamma}^n(\omega) \cdot y > 0\} \quad (11)$$

The inclusion (\supseteq) is clear, for the converse note that if y satisfies $\xi_{t,\Gamma}^n(\omega) \cdot y \leq 0$ for every $n \in \mathbb{N}$ then by continuity $\xi \cdot y \leq 0$ for every $\xi \in \psi_{t,\Gamma}^*(\omega)$.

We now define the conditional standard separator as

$$\xi_{t,\Gamma} := \sum_{n=1}^{\infty} \frac{1}{2^n} \xi_{t,\Gamma}^n \quad (12)$$

which is \mathcal{F}_{t-1}^A -measurable and, from (11), satisfies the following maximality property: $\{\omega \in X \mid \xi(\omega) \cdot \Delta S_t(\omega) > 0\} \subseteq \{\omega \in X \mid \xi_{t,\Gamma}(\omega) \cdot \Delta S_t(\omega) > 0\}$ for any ξ measurable selector of $\psi_{t,\Gamma}^*$.

Step 0: We take $A^0 := \Gamma$ and consider the multifunction ψ_{t,A^0}^* and the conditional standard separator ξ_{t,A^0} in (12). If $\psi_{t,A^0}^*(\omega)$ is a linear subspace of \mathbb{R}^d (i.e. $H \in \psi_{t,A^0}^*(\omega)$ implies necessarily $-H \in \psi_{t,A^0}^*(\omega)$) for any $\omega \in A^0$ then set $\beta = 0$ and $A^0 = B^0$ (In this case obviously $E^0 = X$).

Step 1: If there exists an $\omega \in A^0$ such that $\psi_{t,A^0}^*(\omega)$ is not a linear subspace of \mathbb{R}^d then we set $H_1 = \xi_{t,A^0}$, $E^1 = \{\omega \in X \mid H_1 \Delta S_t > 0\}$, $B^1 = \{\omega \in A^0 \mid H_1 \Delta S_t > 0\} = E^1 \cap \Gamma$ and $A^1 = A^0 \setminus B^1 = \{\omega \in A^0 \mid H_1 \Delta S_t = 0\}$. If now $\psi_{t,A^1}^*(\omega)$ is a linear subspace of \mathbb{R}^d for any $\omega \in A^1$ then we set $\beta = 1$ and $A^1 = B^0$. If this is not the case we proceed iterating this scheme.

Step 2: notice that for every $\omega \in A^1$ we have $\Delta S_t(\omega) \in R_1(\omega) := \{y \in \mathbb{R}^d \mid H_1(\omega) \cdot y = 0\}$ which can be embedded in a subspace of \mathbb{R}^d whose dimension is $d - 1$. We consider the case in which there exists one $\omega \in A^1$ such that $\psi_{t,A^1}^*(\omega)$ is not a linear subspace of $R_1(\omega)$: we set $H_2 = \xi_{t,A^1}$, $E^2 = \{\omega \in X \mid H_2 \Delta S_t > 0\}$, $B^2 = \{\omega \in A^1 \mid H_2 \Delta S_t > 0\} = E^2 \cap \Gamma$ and $A^2 = A^1 \setminus B^2 = \{\omega \in A^1 \mid H_2 \Delta S_t = 0\}$. If now $\psi_{t,A^2}^*(\omega)$ is a linear subspace of $R_1(\omega)$ for any $\omega \in A^2$ then we set $\beta = 2$ and $A^2 = B^0$. If this is not the case we proceed iterating this scheme.

The scheme can be iterated and ends at most within d Steps. ■

Define

$$\begin{aligned} \Omega_T &:= \Omega \\ \Omega_{t-1} &:= \Omega_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \dots, T\}, \end{aligned} \tag{13}$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^\Gamma$ are the sets and index constructed in Lemma 3.1 with $\Gamma = \Omega_t$, for $1 \leq t \leq T$. Note that we can iteratively apply Lemma 3.1 at time $t - 1$ since $\Gamma = \Omega_t$ is \mathcal{F}^A -measurable.

Corollary 3.3 *For any $t \in \{1, \dots, T\}$, $\Omega \in \mathcal{F}^A$ and $Q \in \mathcal{M}_\Omega$ we have $\cup_{i=1}^{\beta_t} B_t^i$ is a subset of a Q -nullset. In particular $\cup_{i=1}^{\beta_t} B_t^i$ is an \mathcal{M}_Ω polar set.*

Proof. If $Q(\Omega_0) = 1$ there is nothing to show. Suppose $Q(\Omega \setminus \Omega_0) > 0$ and, by contradiction, let t be the $\max\{s \in \{1, \dots, T\} \mid Q(\cup_{i=1}^{\beta_s} B_s^i) > 0\}$ so that $\cup_{i=1}^{\beta_t} B_t^i$ is not a Q -nullset, but $\cup_{i=1}^{\beta_s} B_s^i$ is a Q -null set for every $s > t$. This implies $Q(\Omega_t) = 1$. Define $j := \min\{i \in \{1, \dots, \beta_t\} \mid Q(B_t^i) > 0\}$. From Lemma 3.1 item 1, the strategy $H_u := H^j \mathbf{1}_{t-1}(u) \mathbf{1}_{\Omega_t}$ satisfies:

- H is \mathbb{F}^Q -predictable, where $\mathbb{F}^Q = \{\mathcal{F}_t^S \vee \mathcal{N}^Q(\mathcal{B}_X)\}_{t \in \{0, \dots, T\}}$.
- $(H \cdot S)_T \geq 0$ Q -a.s. and $(H \cdot S)_T > 0$ on B_t^j which has positive probability.

Thus, H is an arbitrage in the classical probabilistic sense, which leads to a contradiction. ■

Corollary 3.4 *Let B_t^0 the set provided by Lemma 3.1 for $\Gamma = \Omega_t$. For every $\omega \in B_t^0$ there exists $Q \in \mathcal{P}^f$ with $Q(\{\omega\}) > 0$ such that $\mathbb{E}_Q[S_t \mid \mathcal{F}_{t-1}^S](\omega) = S_{t-1}(\omega)$.*

Proof. Fix $\omega \in B_t^0$ and let Σ_{t-1}^ω be given as in (10). We consider $D := \Delta S_t(\Sigma_{t-1}^\omega \cap B_t^0) \subseteq \mathbb{R}^d$ and $C := \{\lambda v \mid v \in \text{conv}(D), \lambda \in \mathbb{R}^+\}$ where $\text{conv}(D)$ denotes the convex hull of D . Denote by $ri(C)$ the relative interior of C . From Lemma 3.1 item 2 we have $H \cdot \Delta S_t(\tilde{\omega}) \geq 0$ for all $\tilde{\omega} \in \Sigma_{t-1}^\omega \cap B_t^0$ implies $H \cdot \Delta S_t(\tilde{\omega}) = 0$ for all $\tilde{\omega} \in \Sigma_{t-1}^\omega \cap B_t^0$, which is equivalent to $0 \in ri(C)$. From Remark 4.8

in Burzoni et al. (2016b) we have that for every $x \in D$ there exists a finite collection $\{x_j\}_{j=1}^m \subseteq D$ and $\{\lambda_j\}_{j=1}^{m+1}$ with $0 < \lambda_j \leq 1$, $\sum_{j=1}^{m+1} \lambda_j = 1$, such that

$$0 = \sum_{j=1}^m \lambda_j x_j + \lambda_{m+1} x. \quad (14)$$

Choose now $x := \Delta S_t(\omega)$ and note that for every $j = 1, \dots, m$ there exists $\omega_j \in \Sigma_{t-1}^\omega \cap B_t^0$ such that $\Delta S_t(\omega_j) = x_j$. Choose now $Q \in \mathcal{P}^f$ with conditional probability $Q(\cdot \mid \mathcal{F}_{t-1}^S)(\omega) := \sum_{j=1}^m \lambda_j \delta_{\omega_j} + \lambda_{m+1} \delta_\omega$, where $\delta_{\tilde{\omega}}$ denotes the Dirac measure with mass point in $\tilde{\omega}$. From (14), we have the thesis. ■

Lemma 3.5 *The set Ω^* , defined in (4) with $\Phi = 0$, coincides with Ω_0 defined in (13), and therefore $\Omega^* \in \mathcal{F}^A$. Moreover we have the following*

$$\Omega^* \neq \emptyset \iff \mathcal{M}_\Omega \neq \emptyset \iff \mathcal{M}_\Omega^f \neq \emptyset.$$

Proof. The proof is analogous to that of Proposition 4.18 in Burzoni et al. (2016b), but give here a self-contained argument. Notice that $\Omega^* \subseteq \Omega_0$ follows from the definitions and Corollary 3.3. For the reverse inclusion, it suffices to show that for $\omega_* \in \Omega_0$ there exists a $Q \in \mathcal{M}_\Omega$ such that $Q(\{\omega_*\}) > 0$, i.e. $\omega_* \in \Omega^*$. From Corollary 3.4, for any $1 \leq t \leq T$, there exists a finite number of elements of $\Sigma_{t-1}^\omega \cap B_t^0$ named $C_t(\omega) := \{\omega, \omega_1, \dots, \omega_m\}$, such that

$$S_{t-1}(\omega) = \lambda_t(\omega) S_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) S_t(\omega_j) \quad (15)$$

where $\lambda_t(\omega) > 0$ and $\lambda_t(\omega) + \sum_{j=1}^m \lambda_t(\omega_j) = 1$.

Fix now $\omega_* \in \Omega_0$. We iteratively build a set Ω_f^T which is suitable for being the finite support of a discrete martingale measure (and contains ω_*).

Start with $\Omega_f^1 = C_1(\omega_*)$ which satisfies (15) for $t = 1$. For any $t > 1$, given Ω_f^{t-1} , define $\Omega_f^t := \{C_t(\omega) \mid \omega \in \Omega_f^{t-1}\}$. Once Ω_f^T is settled, it is easy to construct a martingale measure via (15):

$$Q(\{\omega\}) = \prod_{t=1}^T \lambda_t(\omega) \quad \forall \omega \in \Omega_f^T$$

Since, by construction, $\lambda_t(\omega_*) > 0$ for any $1 \leq t \leq T$, we have $Q(\{\omega_*\}) > 0$ and $Q \in \mathcal{M}_\Omega^f$. ■

Lemma 3.6 *Suppose $\Phi = 0$. Then, no One-Point Arbitrage $\Leftrightarrow \Omega^* = \Omega$.*

Proof. “ \Leftarrow ” If $(H \circ S)_T \geq 0$ on Ω then $(H \circ S)_T = 0$ Q -a.s. for every $Q \in \mathcal{M}_\Omega^f$. From the hypothesis we have $\cup\{\text{supp}(Q) \mid Q \in \mathcal{M}_\Omega^f\} = \Omega$ from which the thesis follows. “ \Rightarrow ”. Let $1 \leq t \leq T$ and $\Gamma = \Omega_t$. Note that if β_t from Lemma 3.1 is strictly positive then H^1 is a One-Point Arbitrage. We thus have $\beta_t = 0$ for any $1 \leq t \leq T$ and hence $\Omega_0 = \Omega$. From Lemma 3.5 we have $\Omega^* = \Omega$. ■

Definition 3.7 *We call Arbitrage Aggregator the process*

$$H_t^*(\omega) := \sum_{i=1}^{\beta_t} H_t^{i, \Omega_t}(\omega) \mathbf{1}_{B_t^{i, \Omega_t}}(\omega) \quad (16)$$

for $t \in \{1, \dots, T\}$, where $H_t^{i, \Omega_t}, B_t^{i, \Omega_t}, \beta_t$ are provided by Lemma 3.1 with $\Gamma = \Omega_t$.

Remark 3.8 Observe that from Lemma 3.1 item 1, $(H^* \circ S)_T(\omega) \geq 0$ for all $\omega \in \Omega$ and from Lemma 3.5, $(H^* \circ S)_T(\omega) > 0$ for all $\omega \in \Omega \setminus \Omega^*$.

Remark 3.9 By construction we have that H_t^* is \mathcal{F}^A -measurable for every $t \in \{1, \dots, T\}$. Moreover any B_t^{i, Ω_t} is the intersection of an \mathcal{F}_t^A -measurable set with Ω_t . As a consequence we have $(H_t^*)|_{\Omega_t} : \Omega_t \rightarrow \mathbb{R}^d$ is $(\mathcal{F}_t^A)|_{\Omega_t}$ -measurable.

Remark 3.10 In case there are no options to be statically traded, $\Phi = 0$, the enlarged filtration $\tilde{\mathbb{F}}$ required in Theorem 2.3 is given by

$$\tilde{\mathcal{F}}_t : = \mathcal{F}_t^A \vee \sigma(H_1^*, \dots, H_{t+1}^*), \quad t \in \{0, \dots, T-1\} \quad (17)$$

$$\tilde{\mathcal{F}}_T : = \mathcal{F}_T^A \vee \sigma(H_1^*, \dots, H_T^*). \quad (18)$$

so that the Arbitrage Aggregator from (16) is predictable with respect to $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t \in \mathbb{T}}$.

3.2 The case with statically traded options

Definition 3.11 A pathspace partition scheme $\mathcal{R}(\alpha^*, H^*)$ of Ω is a collection of trading strategies $H^1, \dots, H^\beta \in \mathcal{H}(\mathbb{F}^A)$, $\alpha^1, \dots, \alpha^\beta \in \mathbb{R}^k$ and arbitrage aggregators $\tilde{H}^0, \dots, \tilde{H}^\beta$, for some $1 \leq \beta \leq k$, such that

(i) α^i , $1 \leq i \leq \beta$, are linearly independent,

(ii) for any $i \leq \beta$,

$$(H^i \circ S)_T + \alpha^i \cdot \Phi \geq 0 \text{ on } A_{i-1}^*,$$

where $A_0 = \Omega$, $A_i := \{(H^i \circ S)_T + \alpha^i \cdot \Phi = 0\} \cap A_{i-1}^*$ and A_i^* is the set Ω^* in (4) with $\Omega = A_i$ and $\Phi = 0$ for $1 \leq i \leq \beta$,

(iii) for any $i = 0, \dots, \beta$, \tilde{H}^i is the Arbitrage Aggregator, as defined in (16) substituting Ω with A_i ,

(iv) if $\beta < k$, then either $A_\beta^* = \emptyset$, or for any $\alpha \in \mathbb{R}^k$ linearly independent from $\alpha^1, \dots, \alpha^\beta$, there does not exist H such that

$$(H \circ S)_T + \alpha \cdot \Phi \geq 0 \text{ on } A_\beta^*.$$

We note that as defined in (ii) above, each $A_i \in \mathcal{F}^A$ so that A_i^* in (iii) is well defined. The purpose of a pathspace partition scheme is to iteratively split the pathspace Ω in subsets on which a Strong Arbitrage strategy can be identified. For the existence of calibrated martingale measure it will be crucial to see whether this procedure exhausts the pathspace or not. Note that on A_i we can perfectly replicate i linearly independent combinations of options $\alpha^j \cdot \Phi$, $1 \leq j \leq i$. In consequence, we make at most k such iterations, $\beta \leq k$, and if $\beta = k$ then all statically traded options are perfectly replicated on A_β^* which reduces here to the setting without statically traded options.

Definition 3.12 A pathspace partition scheme $\mathcal{R}(\alpha^*, H^*)$ is successful if $A_\beta^* \neq \emptyset$.

We illustrate now the construction of a successful pathspace partition scheme.

Example 3.13 Let $X = \mathbb{R}^2$. Consider a financial market with one dynamically traded asset S and two options available for static trading $\Phi := (\phi_1, \phi_2)$. Let S be the canonical process i.e. $S_t(x) = x_t$ for $t = 1, 2$ with initial price $S_0 = 2$. Moreover, let $\phi_i := g_i(S_1, S_2) - c$ for $i = 1, 2$ with $c > 0$, $g_i := (x_2 - K_i)^+ \mathbf{1}_{[0, b]}(x_1) + c \mathbf{1}_{(b, \infty)}(x_1)$ and $K_1 > K_2$. Namely, ϕ_i is a knock in call option on S with maturity $T = 2$, strike price K_i , knock in value $b \geq 0$ and cost $c > 0$, but the cost is recovered if the option is not knocked in. Consider the pathspace selection $\Omega = [0, 4] \times [0, 4]$.

Start with $A_0 := \Omega$ and suppose $0 < K_2 < b < 2$.

1. the process $(\tilde{H}_1^0, \tilde{H}_2^0) := (0, \mathbf{1}_{\{0\}}(S_1) - \mathbf{1}_{\{4\}}(S_1))$ is an arbitrage aggregator: when S_1 hits the values $\{0, 4\}$, the price process does not decrease, or increase, respectively. It is easily seen that there are no more arbitrages on A_0 from dynamic trading only. Thus, $A_0^* = \{(0, 4) \times [0, 4]\} \cup \{\{0\} \times \{0\}\} \cup \{\{4\} \times \{4\}\}$.
2. Suppose now we have a semi-static strategy (H_1, α^1) such that

$$(H^1 \circ S)_T + \alpha^1 \cdot \Phi \geq 0 \text{ on } A_0^*.$$

where $\alpha^1 = (-\alpha, \alpha) \in \mathbb{R}^2$ for some $\alpha > 0$ (since $K_1 > K_2$ and ϕ_1, ϕ_2 have the same cost). Moreover since $\Phi = 0$ if $S_1 \notin [0, b]$, we can choose $(H_1^1, H_2^1) = (0, 0)$. A positive gain is obtained on $B_1 := [0, b] \times (K_2, 4]$ (see Figure 1a). Thus, $A_1 := A_0^* \setminus B_1$.

3. $(\tilde{H}_1^1, \tilde{H}_2^1) := (0, -\mathbf{1}_{[K_2, b]}(S_1))$ is an arbitrage aggregator on A_1 : the price process does not increase and $(\tilde{H}_1^1, \tilde{H}_2^1)$ yields a positive gain on $B_2 := \{K_2 \leq S_1 \leq b\} \setminus \{\{K_2\} \times \{K_2\}\}$ (see Figure 1b). Thus, $A_1^* = A_1 \setminus B_2$.
4. The null process H^2 and the vector $\alpha^2 := (-1, 0) \in \mathbb{R}^2$ satisfies

$$(H^2 \circ S)_T + \alpha^2 \cdot \Phi \geq 0 \text{ on } A_1^*.$$

A positive gain is obtained on $[0, b] \times [0, K_2]$. Thus $A_2 := (b, 4) \times [0, 4] \cup \{\{4\} \times \{4\}\}$.

Obviously there are no more semi-static 1p-arbitrage opportunities and $A_2 = A_2^*$, $\beta = 2$. We set $\tilde{H}^2 \equiv 0$ and the partition scheme is successful with arbitrage aggregators $\tilde{H}^0, \tilde{H}^1, \tilde{H}^2$, and semi-static strategies (H^j, α^j) , for $j = 1, 2$, as above.

Remark 3.14 The previous example also shows that $\pi_{\Omega_\Phi^*, \Phi}(g) = \pi_{\Omega, \Phi}(g)$ is a rather exceptional case if we do not assume the existence of an option with dominating payoff as in Theorem 2.9. Consider indeed the market of example 3.13 with $b := 4$, $0 < K_2 < 2$, which has the same features as the example on page 5 in Davis and Hobson (2007). Take $g \equiv 1$ and note that Ω is compact and S, Φ and g are continuous functions on Ω . From the above discussion we see easily that $\Omega_\Phi^* = \emptyset$ and $\pi_{\Omega_\Phi^*} = -\infty$. Nevertheless, by considering the pathspace partition scheme above, we see that while we can devise an arbitrage strategy on $B_1 = [0, 4) \times (K_2, 4]$ its payoff is not bounded below by a positive constant and in fact we see that $\pi_{\Omega, \Phi}(g) = 1$.

Remark 3.15 Note that if a partition scheme is successful then there are no One-Point Arbitrages on A_β^* . When $\beta < k$ this follows from (iv) in Definition 3.11. In the case $\beta = k$ suppose there is

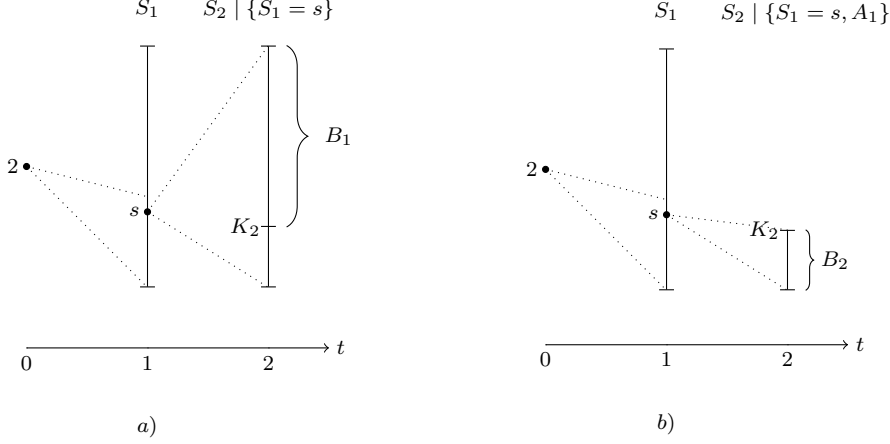


Figure 1: Some steps of the pathspace partition scheme with $b = 1.5$, $K_2 = 1$. On the left, the strategy (H_1, α^1) , defined on A_0^* , has positive gain on B_1 . As a consequence the pathspace reduces to $A_1 = A_0^* \setminus B_1$. On the right, for $S_1 \geq K_2$ the arbitrage aggregator \tilde{H}_1 , defined on A_1 , has positive gain on B_2 .

a One-Point Arbitrage $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^A)$ so that, in particular, $(H \circ S)_T + \alpha \cdot \Phi \geq 0$ on A_β^* . Since the vectors α^i form a basis of \mathbb{R}^k we get, for some $\lambda_i \in \mathbb{R}$,

$$(H \circ S)_T + \alpha \cdot \Phi = \sum_{i=1}^k \lambda_i [(H^i \circ S)_T + \alpha^i \cdot \Phi] + (\hat{H} \circ S)_T$$

where $\hat{H} := H - \sum_{i=1}^k \lambda_i H^i$. Since, by construction $(H^i \circ S)_T + \alpha^i \cdot \Phi = 0$ on A_β^* for any $i = 1, \dots, \beta$, we obtain that \hat{H} is a One-Point Arbitrage with $\Phi = 0$ on A_β^* . From Lemma 3.6 we have a contradiction.

The following Lemma shows that the outcome of a successful partition scheme is the set Ω_Φ^* .

Lemma 3.16 Recall the definition of Ω_Φ^* in (4). For any $\mathcal{R}(\alpha^*, H^*)$, $A_i^* = \Omega_{\{\alpha^j \cdot \Phi : j \leq i\}}^*$ for any $i \leq \beta$. Moreover, if $\mathcal{R}(\alpha^*, H^*)$ is successful, then $A_\beta^* = \Omega_\Phi^*$.

Proof. If $\Omega^* = \emptyset$, then the claim holds trivial. We now assume $\Omega^* \neq \emptyset$, fix a partition scheme $\mathcal{R}(\alpha^*, H^*)$ and prove the claim by induction on i . For simplicity of notation, let $\Omega_i^* := \Omega_{\{\alpha^j \cdot \Phi : j \leq i\}}^*$ with $\Omega_0 = \Omega$. By definition of A_0 we have $A_0^* = \Omega^* = \Omega_0^*$. Suppose now $A_{i-1}^* = \Omega_{i-1}^*$ for some $i \leq \beta$. Then, by definition of Ω_i we have, $\Omega_i^* \subset \Omega_{i-1}^* = A_{i-1}^*$. Further, since $(H_i \circ S)_T + \alpha^i \cdot \Phi \geq 0$ on A_{i-1}^* with strict inequality on $A_{i-1}^* \setminus A_i$, it follows that $\Omega_i^* \subset A_i$. Finally, from $\mathcal{M}_{\Omega_i^*, \{\alpha^j \cdot \Phi : j \leq i\}}^f \subset \mathcal{M}_{A_i, \{\alpha^j \cdot \Phi : j \leq i\}}^f \subset \mathcal{M}_{A_i}^f = \mathcal{M}_{A_i^*}^f$ we also have $\Omega_i^* \subset A_i^*$. For the reverse inclusion consider $\omega \in A_i^*$. By definition of A_i^* and Lemma 3.5, there exists $Q \in \mathcal{M}_{A_i^*}^f$ with $Q(\{\omega\}) > 0$. Since on A_i^* , all options $\alpha^j \cdot \Phi$, $1 \leq j \leq i$, are perfectly replicated by the dynamic strategies $-H^j$, it follows that $Q \in \mathcal{M}_{\Omega_i^*, \{\alpha^j \cdot \Phi : j \leq i\}}^f$ so that $\omega \in \Omega_i^*$.

Suppose now $\mathcal{R}(\alpha^*, H^*)$ is successful. In the case $\beta = k$, since α^i form a basis of \mathbb{R}^k we have $\mathcal{M}_{A_\beta^*, \Phi}^f = \mathcal{M}_{A_\beta^*, \{\alpha^j \cdot \Phi : j \leq \beta\}}^f$ and hence $\Omega_\Phi^* = A_\beta^*$ from the above. Suppose $\beta < k$ so that the above shows $\Omega_\Phi^* \subset \Omega_\beta^* = A_\beta^*$. Observe that since $A_\beta \in \mathcal{F}^A$, from Lemma 3.5, we have $A_\beta^* \in \mathcal{F}^A$, moreover, by Definition 3.11 (iv), there are no One-Point Arbitrages on A_β^* . Thus, by Proposition 2.6, we have $\Omega_\Phi^* = A_\beta^*$ as required. ■

Remark 3.17 The above lemma implies relative uniqueness of $\mathcal{R}(\alpha^*, H^*)$ in the sense that either every $\mathcal{R}(\alpha^*, H^*)$ is not successful or all $\mathcal{R}(\alpha^*, H^*)$ are successful and then $A_\beta^* = \Omega_\Phi^*$. It follows also from Lemma 3.5 that $A_i^* \equiv \Omega_{\{\alpha^j, \Phi: j \leq i\}}^*$ belongs to \mathcal{F}^A for any $i \leq \beta$. In particular Ω_Φ^* in (4) is in \mathcal{F}^A .

Definition 3.18 Given a pathspace partition scheme we define the Arbitrage Aggregator as

$$(\alpha^*, H^*) = \left(\sum_{i=1}^{\beta} \alpha^i \mathbf{1}_{A_i^*}, \sum_{i=1}^{\beta} H^i \mathbf{1}_{A_i^*} + \sum_{i=1}^{\beta} \tilde{H}^i \mathbf{1}_{A_i \setminus A_i^*} \right), \quad (19)$$

with $(\alpha^*, H^*) = (0, \tilde{H}^0 \mathbf{1}_{\Omega \setminus \Omega^*})$ if $\beta = 0$.

To make the above arbitrage aggregator predictable we need to enlarge the filtration. We therefore introduce the arbitrage aggregating filtration $\tilde{\mathbb{F}}$ given by

$$\begin{aligned} \tilde{\mathcal{F}}_t &= \mathcal{F}_t^A \vee \{A_0, A_0^*, \dots, A_\beta, A_\beta^*\} \vee \sigma(\tilde{H}_1^0, \dots, \tilde{H}_1^\beta, \dots, \tilde{H}_{t+1}^0, \dots, \tilde{H}_{t+1}^\beta), \\ \tilde{\mathcal{F}}_T &= \mathcal{F}_T^A \vee \{A_0, A_0^*, \dots, A_\beta, A_\beta^*\} \vee \sigma(\tilde{H}_1^0, \dots, \tilde{H}_1^\beta, \dots, \tilde{H}_T^0, \dots, \tilde{H}_T^\beta). \end{aligned} \quad (20)$$

Note that $\tilde{\mathcal{F}}_t \subset \mathcal{F}_T^A$ for any $t = 0, \dots, T$ and in particular, as observed before, any $\mathbb{Q} \in \mathcal{M}_{\Omega, \Phi}(\mathbb{F}^S)$ extends uniquely to a measure in $\mathcal{M}_{\Omega, \Phi}^f(\tilde{\mathbb{F}})$.

4 Proofs

We first describe the logical flow of our proofs and we point out that showing $\Omega_\Phi^* \in \mathcal{F}^A$ is involved. First, Theorem 2.3 and then Theorem 2.5 are established when $\Phi = 0$. Then, we show Theorem 2.5 *under the further assumption* that $\Omega_{\Phi_i}^* \in \mathcal{F}^A$ for all $1 \leq i \leq k$, where $\Phi_i = \{\phi_1, \dots, \phi_i\}$. Note that in the case with no statically traded options ($\Phi = 0$) the property $\Omega_\Phi^* = \Omega^* \in \mathcal{F}^A$ follows from the construction and is shown in Lemma 3.5. This allows us to prove Proposition 2.6 for which we use Theorem 2.5 only when $\Omega_{\Phi_i}^* = \Omega$, which belongs to \mathcal{F}^A by assumption. This proof is presented directly below. Proposition 2.6 in turn allows us to establish Lemma 3.16 which implies that in all cases $\Omega_{\Phi_i}^* \in \mathcal{F}^A$. This then completes the proofs of Theorem 2.3 and Theorem 2.5 in the general setting.

Proof of Proposition 2.6. “ \Leftarrow ” is clear since, if a strategy $(\alpha, H) \in \mathcal{A}_\Phi(\mathbb{F}^A)$ satisfies $\alpha \cdot \Phi + (H \circ S)_T \geq 0$ on Ω then, by definition in (4), for any $\omega \in \Omega_\Phi^* = \Omega$, we can take a calibrated martingale measure which assigns a positive probability to ω , which implies $\alpha \cdot \Phi + (H \circ S)_T = 0$ on ω . Since ω is arbitrary we obtain the thesis. We prove “ \Rightarrow ” by iteration on number of options used for static trading. No One-Point Arbitrage using dynamic trading and Φ in particular means that there is no One-Point Arbitrage using only dynamic trading. From Lemma 3.6 we have $\Omega^* = \Omega$ and hence for any $\omega \in \Omega$ there exists $Q \in \mathcal{M}_\Omega^f$ such that $Q(\{\omega\}) > 0$.

Note that if, for some $j \leq k$, ϕ_j is replicable on Ω^* by dynamic trading in S then there exists $(x, H) \in \mathbb{R} \times \mathcal{H}(\mathbb{F}^A)$ such that $x + (H \circ S)_T = \phi_j$ on Ω^* . No One-Point Arbitrage implies $x = 0$ and hence $E_Q[\phi_j] = 0$ for every $Q \in \mathcal{M}_\Omega^f$. With no loss of generality we assume that $(\phi_1, \dots, \phi_{k_1})$ is a vector of non-replicable options on Ω^* with $k_1 \leq k$. We now apply Theorem 2.5 in the case with $\Phi = 0$ to ϕ_1 and argue that

$$m_1 := \min\{\pi_{\Omega^*}(\phi_1), \pi_{\Omega^*}(-\phi_1)\} > 0.$$

Indeed, if $m_1 < 0$ then we would have a Strong Arbitrage and if $m_1 = 0$, since the superhedging price is attained, there exists $H \in \mathcal{H}(\mathbb{F}^A)$ such that $\phi_1 \leq (H \circ S)_T$ on Ω . In order to avoid One-Point Arbitrage we have to have $\phi_1 = (H \circ S)_T$ on Ω which is a contradiction since ϕ_1 is not replicable. This shows that $m_1 > 0$ which in turn implies there exist $Q_1, Q_2 \in \mathcal{M}_\Omega^f$ such that $E_{Q_1}[\phi_1] > 0$ and $E_{Q_2}[\phi_1] < 0$. Then, for any $Q \in \mathcal{M}_\Omega^f$, there exist $\alpha, \beta, \gamma \in [0, 1]$, $\alpha + \beta + \gamma = 1$ and $E_{\alpha Q_1 + \beta Q_2 + \gamma Q}[\phi_1] = 0$. Thus, for any $\omega \in \Omega^*$ there exists $Q \in \mathcal{M}_{\Omega, \phi_1}^f$ such that $Q(\{\omega\}) > 0$. In particular, $\Omega_{\phi_1}^* = \Omega$ and we may apply Theorem 2.5 with Ω and $\Phi = \{\phi_1\}$. Define now

$$m_{1,j} := \min\{\pi_{\Omega^*, \phi_1}(\phi_j), \pi_{\Omega^*, \phi_1}(-\phi_j)\} \quad \forall j = 2, \dots, k_1.$$

By absence of Strong Arbitrage we necessarily have $m_{1,j} \geq 0$ for every $j = 2, \dots, k_1$. Let $j \in I_2 = \{j = 2, \dots, k_1 \mid m_{1,j} = 0\}$, by No One-Point Arbitrage, we have perfect replication of ϕ_j using semistatic strategies with ϕ_1 on Ω and in consequence for any $Q \in \mathcal{M}_{\Omega, \phi_1}^f$ we have $E_Q[\phi_j] = 0$ for all $j \in I_2$. We may discard these options and, up to re-numbering, assume that $(\phi_2, \dots, \phi_{k_2})$ is a vector of the remaining options, non-replicable on Ω with semistatic trading in ϕ_1 , with $k_2 \leq k_1$. If $k_2 \geq 2$, $m_{1,2} > 0$ by Theorem 2.5 and absence of One-Point Arbitrage using arguments as above. Hence, there exist $Q_1, Q_2 \in \mathcal{M}_{\Omega, \phi_1}^f$ such that $E_{Q_1}[\phi_2] > 0$ and $E_{Q_2}[\phi_2] < 0$. As above, this implies that $\Omega_{\{\phi_1, \phi_2\}}^* = \Omega_{\phi_1}^* = \Omega$. We can iterate the above arguments and the procedure ends after at most k steps showing $\Omega_\Phi^* = \Omega$ as required. ■

4.1 Proof of the Pointwise FTAP

Proof of Theorem 2.3, when no options are statically traded. In this case we consider the technical filtration as described in Remark 3.10 and the Arbitrage Aggregator H^* defined by (16). We prove that

$$\exists \text{ Strong Arbitrage on } \Omega \text{ in } \mathcal{H}(\widetilde{\mathbb{F}}) \Leftrightarrow \mathcal{M}_\Omega = \emptyset.$$

Notice that if $H \in \mathcal{H}(\widetilde{\mathbb{F}})$ satisfies $(H \circ S)_T(\omega) > 0 \forall \omega \in \Omega$ then, if there exists $Q \in \mathcal{M}_\Omega$ we would get $0 < \mathbb{E}_Q[(H \circ S)_T] = 0$ which is a contradiction. For the opposite implication, let H^* be the Arbitrage Aggregator from (16) and note that $(H^* \circ S)_T(\omega) \geq 0 \forall \omega \in \Omega$ and $\{\omega \mid (H^* \circ S)_T(\omega) > 0\} = (\Omega^*)^c$. If $\mathcal{M}_\Omega = \emptyset$ then, by Lemma 3.5, $(\Omega^*)^c = \Omega$ and H^* is therefore a Strong Arbitrage on Ω in $\mathcal{H}(\widetilde{\mathbb{F}})$. The last assertion, namely $\Omega^* = \{\omega \in \Omega \mid (H^* \circ S)_T(\omega) = 0\}$, follows straightforwardly from the definition of H^* . ■

We now prove the pathwise Fundamental Theorem of Asset Pricing, when semistatic trading strategies are allowed.

Proof of Theorem 2.3, with semistatic strategies. Let $\Omega \in \mathcal{F}^A$, $\mathcal{R}(\alpha^*, H^*)$ be a pathspace partition scheme and $\widetilde{\mathbb{F}}$ be given by (20). We first show that the following are equivalent:

1. $\mathcal{R}(\alpha^*, H^*)$ is successful.
2. $\mathcal{M}_{\Omega, \Phi}^f \neq \emptyset$.
3. $\mathcal{M}_{\Omega, \Phi} \neq \emptyset$.
4. No Strong Arbitrage with respect to $\widetilde{\mathbb{F}}$.

By Remark 3.17, $A_\beta^* = \Omega_\Phi^*$ and hence (1) \Leftrightarrow (2) follows by definition of Ω_Φ^* in (4). (2) \Rightarrow (3) is obvious. To show (3) \Rightarrow (4), observe that under $Q \in \mathcal{M}_{\Omega, \Phi}^f$ the expectation of any admissible semi-static trading strategy is zero which excludes the possibility of existence of a Strong Arbitrage. For the implication (4) \Rightarrow (1) note that, for $1 \leq i \leq \beta$, we have $(\tilde{H}^{i-1} \circ S)_T > 0$ on $A_{i-1} \setminus A_{i-1}^*$ from the properties of the Arbitrage Aggregator (see Remark 3.8) and $(H^i \circ S)_T + \alpha^i \cdot \Phi > 0$ on $A_{i-1}^* \setminus A_i$ by construction, so that a positive gain is realized on $A_{i-1} \setminus A_i$. Finally, from $\Omega = A_0 = (\cup_{i=1}^\beta A_{i-1} \setminus A_i) \cup A_\beta$ and $(\tilde{H}^\beta \circ S)_T > 0$ on $A_\beta \setminus A_\beta^*$, we get

$$\sum_{i=1}^\beta (H^i \circ S)_T + \alpha^i \cdot \Phi + \sum_{i=0}^\beta (\tilde{H}^i \circ S)_T > 0 \quad \text{on } (A_\beta^*)^C, \quad (21)$$

and equal to 0 otherwise. The hypothesis (4) implies therefore that A_β^* is non-empty and hence the pathspace partition scheme is successful.

The existence of the technical filtration and Arbitrage Aggregator are provided explicitly by (19) and (20). Moreover, from Lemma 3.16, $\Omega_\Phi^* = A_\beta^* \in \mathcal{F}^A$. Finally, equation (5) follows from (21). \blacksquare

4.2 Proof of Theorem 2.5

Proposition 4.1 (Superhedging on $\Omega \subseteq X$ without options) *Let $\Omega \in \mathcal{F}^A$. We have that for any $g \in \mathcal{L}(X, \mathcal{F}^A; \mathbb{R})$*

$$\pi_{\Omega^*}(g) = \sup_{Q \in \mathcal{M}_\Omega^f} E_Q[g]. \quad (22)$$

with $\pi_{\Omega^}(g) = \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H}(\mathbb{F}^A) \text{ such that } x + (H \circ S)_T \geq g \text{ on } \Omega^*\}$. In particular, the left hand side of (22) is attained by some strategy $H \in \mathcal{H}(\mathbb{F}^A)$.*

Proof. Note that by its definition in (4), $\Omega^* = \emptyset$ if and only if $\mathcal{M}_\Omega^f = \emptyset$ and in this case both sides in (22) are equal to $-\infty$. We assume now that $\Omega^* \neq \emptyset$ and recall from Lemma 3.5 we have $\Omega^* \in \mathcal{F}^A$. The second part of the statement follows with the same procedure proposed in Burzoni et al. (2016a) proof of Theorem 1.1. The reason can be easily understood recalling the following construction, which appears in Step 1 of the proof. For any $D \in \mathcal{F}^A$, $1 \leq t \leq T$, $X \in \mathcal{L}(X, \mathcal{F}^A)$, we define the multifunction

$$\psi_{t,X,D} : \omega \mapsto \{[\Delta S_t(\tilde{\omega}); 1; X(\tilde{\omega})] \mathbf{1}_D(\tilde{\omega}) \mid \tilde{\omega} \in \Sigma_{t-1}^\omega\} \subseteq \mathbb{R}^{d+2}$$

where $[\Delta S_t; 1; X] \mathbf{1}_D = [\Delta S_t^1 \mathbf{1}_D, \dots, \Delta S_t^d \mathbf{1}_D, \mathbf{1}_D, X \mathbf{1}_D]$ and Σ_t^ω is given as in (10). We show that $\psi_{t,X,D}$ is an \mathcal{F}_{t-1}^A -measurable multifunction. Indeed we need to show that, for any open set $O \subseteq \mathbb{R}^d \times \mathbb{R}^2$,

$$\{\omega \in X \mid \psi_{t,X,D}(\omega) \cap O \neq \emptyset\} = (S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)),$$

where $B = ([\Delta S_t; 1; X] \mathbf{1}_D)^{-1}(O)$. First $[\Delta S_t, 1, X] \mathbf{1}_D$ is an \mathcal{F}^A -measurable random vector then $B \in \mathcal{F}^A$. Second S_u, Y_u is a Borel measurable function for any $0 \leq u \leq t-1$ so that we have, as a consequence of Lemma 5.1, that $(S_{0:t-1}, Y_{0:t-1})(B)$ belongs to the sigma-algebra generated by the analytic sets in $Mat((d + d_n) \times t; \mathbb{R})$ endowed with its Borel sigma-algebra. Applying again Lemma 5.1 we deduce that $(S_{0:t-1}, Y_{0:t-1})^{-1}((S_{0:t-1}, Y_{0:t-1})(B)) \in \mathcal{F}_{t-1}^A$ and hence the desired

measurability for $\psi_{t,X,D}$.

The remaining of Step 1, Step 2, Step 3, Step 4 and Step 5 follows replicating the argument in Burzoni et al. (2016a). ■

Proof of Theorem 2.5. Similarly to the previous proof, we note that the statement is clear when $\mathcal{M}_{\Omega, \Phi}^f = \emptyset$ so we may assume to the contrary. Recall that $\Phi_n = (\phi_1, \dots, \phi_n)$, $1 \leq n \leq k$ with $\Phi_k = \Phi$. We prove the statement by induction on the number of static options used for superhedging. For this we consider the superhedging problem with additional options Φ_n on Ω_Φ^* and denote its superhedging cost by $\pi_{\Omega_\Phi^*, \Phi_n}(g)$ which is defined as in (6) but with Φ_n replacing Φ . We first assume that $\Omega_{\Phi_n}^* \in \mathcal{F}^A$ for all $n \leq k$. This will be justified at the end. The case $n = 0$ corresponds to the super-hedging problem on Ω^* when only dynamic trading is possible. Since by assumption $\Omega_\Phi^* \in \mathcal{F}^A$, the pricing–hedging duality and the attainment of the infimum follow from Proposition 4.1. Now assume that for some $n < k$, for any \mathcal{F}^A -measurable g , we have the following pricing–hedging duality

$$\pi_{\Omega_\Phi^*, \Phi_n}(g) = \sup_{Q \in \mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f} E_Q[g] \quad (23)$$

We show that the same statement holds for $n + 1$. Note that the attainment property is always satisfied. Indeed, as in the proof of Proposition 2.6 (\Leftarrow), we have that No One-point Arbitrage holds on Ω_Φ^* . In particular, this implies $NA(\mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f)$, in the notation of Bouchard and Nutz (2015) and by Theorem 2.3 in Bouchard and Nutz (2015), which holds in also in the setup of this paper, the infimum is attained whenever is finite.

The proof proceeds in three steps.

Step 1. First observe that if ϕ_{n+1} is replicable on Ω_Φ^* by semi-static portfolios with the static hedging part restricted to Φ_n , i.e. $x + h \cdot \Phi_n(\omega) + (H \circ S)_T(\omega) = \phi_{n+1}(\omega)$, for any $\omega \in \Omega_\Phi^*$, then necessarily $x = 0$ (otherwise $\mathcal{M}_{\Omega, \Phi}^f = \emptyset$). Moreover since any such portfolio has zero expectation under measures in $\mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f$ we have that $E_Q[\phi_{n+1}] = 0 \ \forall Q \in \mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f$. In particular $\mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f = \mathcal{M}_{\Omega_\Phi^*, \Phi_{n+1}}^f$ and (23) holds for $n + 1$.

Step 2. We now look at the more interesting case, that is ϕ_{n+1} is not replicable. In this case, we show that:

$$\sup_{Q \in \mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f} E_Q[\phi_{n+1}] > 0 \quad \text{and} \quad \inf_{Q \in \mathcal{M}_{\Omega_\Phi^*, \Phi_n}^f} E_Q[\phi_{n+1}] < 0. \quad (24)$$

Inequalities \geq and \leq are obvious from the assumption $\mathcal{M}_{\Omega_\Phi^*, \Phi}^f \neq \emptyset$. From the inductive hypothesis we only need to show that $\pi_{\Omega_\Phi^*, \Phi_n}(\phi_{n+1})$ is always strictly positive (analogous argument applies to $\pi_{\Omega_\Phi^*, \Phi_n}(-\phi_{n+1})$). Suppose, by contradiction, $\pi_{\Omega_\Phi^*, \Phi_n}(\phi_{n+1}) = 0$. Since the infimum is attained, there exists some $(\alpha, H) \in \mathbb{R}^n \times \mathcal{H}(\mathbb{F}^A)$ such that

$$\alpha \cdot \Phi_n(\omega) + (H \circ S)_T(\omega) \geq \phi_{n+1}(\omega) \quad \forall \omega \in \Omega_\Phi^*.$$

Since ϕ_{n+1} is not replicable the above inequality is strict for some $\tilde{\omega} \in \Omega_\Phi^*$. Then, by taking expectation under $\tilde{Q} \in \mathcal{M}_{\Omega_\Phi^*, \Phi}^f$ such that $\tilde{Q}(\{\tilde{\omega}\}) > 0$, we obtain

$$0 = E_{\tilde{Q}}[\alpha \cdot \Phi_n + (H \circ S)_T] > E_{\tilde{Q}}[\phi_{n+1}] = 0. \quad (25)$$

which is clearly a contradiction.

Step 3. Given (24), we now show that (23) holds for $n + 1$, also in the case that ϕ_{n+1} is not replicable. We first use a variational argument to deduce the following equalities:

$$\begin{aligned}
\pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) &= \inf_{l \in \mathbb{R}} \pi_{\Omega_{\Phi}^*, \Phi_n}(g - l\phi_{n+1}) \\
&= \inf_{l \in \mathbb{R}} \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} E_Q[g - l\phi_{n+1}] \\
&= \inf_N \inf_{|l| \leq N} \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} E_Q[g - l\phi_{n+1}] \\
&= \inf_N \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} \inf_{|l| \leq N} E_Q[g - l\phi_{n+1}], \\
&= \inf_N \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} (E_Q[g] - N|E_Q[\phi_{n+1}]|)
\end{aligned} \tag{26}$$

The first equality follows by definition, the second from the inductive hypothesis, the fourth is obtained with an application of min-max theorem (see Corollary 2 in Terkelsen (1972)) and the last one follows from an easy calculation.

We also observe that there exist $Q_{\sup} \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f$ and $Q_{\inf} \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f$ such that

$$E_{Q_{\sup}}[\phi_{n+1}] \geq \frac{1}{2} (\pi_{\Omega_{\Phi}^*, \Phi_n}(\phi_{n+1}) \wedge 1) \quad \text{and} \quad E_{Q_{\inf}}[\phi_{n+1}] \leq -\frac{1}{2} (\pi_{\Omega_{\Phi}^*, \Phi_n}(-\phi_{n+1}) \wedge 1).$$

From (24) and the inductive hypothesis $E_{Q_{\inf}}[\phi_{n+1}] < 0 < E_{Q_{\sup}}[\phi_{n+1}]$. We will later use Q_{\inf} and Q_{\sup} for calibrating measures in $\mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f$ to the additional option ϕ_{n+1} . Namely, for $Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f$ we might set $\tilde{Q} = Q_{\inf}$ if $E_Q[\phi_{n+1}] \geq 0$, and Q_{\sup} otherwise, to find $\lambda \in [0, 1]$ such that $\hat{Q} = (1 - \lambda)Q + \lambda\tilde{Q} \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f$.

We can now distinguish two cases:

Case 1. Suppose first there exists a sequence $\{Q_m\} \subseteq \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f \setminus \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f$ such that

$$\lim_{m \rightarrow \infty} \frac{E_{Q_m}[g]}{|E_{Q_m}[\phi_{n+1}]|} = +\infty \quad \text{and} \quad \lim_{m \rightarrow \infty} E_{Q_m}[g] = +\infty. \tag{27}$$

Given $\{Q_m\}$ such that (27) is satisfied, we can construct a sequence of calibrated measures $\{\hat{Q}_m\} \subseteq \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f$, as described above, so that

$$E_{\hat{Q}_m}[\phi_{n+1}] = \lambda_m E_{Q_m}[\phi_{n+1}] + (1 - \lambda_m) E_{\tilde{Q}_m}[\phi_{n+1}] = 0,$$

for some $\{\lambda_m\} \subseteq [0, 1]$. We stress that \tilde{Q}_m can only be equal to Q_{\inf} or Q_{\sup} , which do not depend on m . A simple calculation shows

$$\lambda_m = \frac{E_{\tilde{Q}_m}[\phi_{n+1}]}{E_{\tilde{Q}_m}[\phi_{n+1}] - E_{Q_m}[\phi_{n+1}]}.$$

From

$$E_{\hat{Q}_m}[g] = \lambda_m(E_{Q_m}[g] - E_{\tilde{Q}_m}[g]) + E_{\tilde{Q}_m}[g]$$

we have two cases: either $\lambda_m \rightarrow a > 0$ and from $E_{Q_m}[g] \rightarrow +\infty$ we deduce $E_{\hat{Q}_m}[g] \rightarrow +\infty$; or $\lambda_m \rightarrow 0$ which happens when $|E_{Q_m}[\phi_{n+1}]| \rightarrow \infty$. Nevertheless in such a case, from (27) we obtain again $E_{\hat{Q}_m}[g] \rightarrow +\infty$ as $m \rightarrow \infty$. Therefore, $\infty = \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f} E_Q[g] \leq \pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g)$ and hence the duality.

Case 2 We are only left with the case where (27) is not satisfied. For any $N \in \mathbb{N}$, we define the decreasing sequence $s_N := \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} (E_Q[g] - N|E_Q[\phi_{n+1}]|)$ and let $\{Q_N^m\}_{m \in \mathbb{N}}$ a sequence realizing the supremum. If there exists a subsequence s_{N_j} such that $|E_{Q_{N_j}^m}[\phi_{n+1}]| = 0$ for $m > \bar{m}(N_j)$, then the duality follows directly from (26). Suppose this is not the case. We claim that we can find a sequence $\{Q_N\} \subseteq \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f$ such that

$$\lim_{N \rightarrow \infty} (E_{Q_N}[g] - N|E_{Q_N}[\phi_{n+1}]|) = \lim_{N \rightarrow \infty} s_N \quad \text{and} \quad \lim_{N \rightarrow \infty} |E_{Q_N}[\phi_{n+1}]| = 0. \quad (28)$$

Let indeed $c(N) := \limsup_{m \rightarrow \infty} E_{Q_N^m}[g]/|E_{Q_N^m}[\phi_{n+1}]|$. If $\sup_{N \in \mathbb{N}} c(N) = \infty$, since (27) is not satisfied, there exists $m = m(N)$ such that $|E_{Q_N^{m(N)}}[\phi_{n+1}]|$ converges to 0 as $N \rightarrow \infty$, from which the claim easily follows. Suppose now $\sup_{N \in \mathbb{N}} c(N) < \infty$. Then (by taking subsequences if needed),

$$s_N = \lim_{m \rightarrow \infty} (E_{Q_N^m}[g] - N|E_{Q_N^m}[\phi_{n+1}]|) \leq (c(N) - N) \lim_{m \rightarrow \infty} |E_{Q_N^m}[\phi_{n+1}]|,$$

Note that $a_N := \lim_{m \rightarrow \infty} |E_{Q_N^m}[\phi_{n+1}]|$ satisfies $\lim_{N \rightarrow \infty} Na_N < \infty$ otherwise, from (26), $\pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) = -\infty$ which is not possible as, for any $Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi}^f$, we have that $\pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) \geq E_Q[g] > -\infty$. In particular, $\lim_{N \rightarrow \infty} a_N = 0$ and the claim easily follows.

Given a sequence as in (28) we now conclude the proof. It follows from (26) that

$$\begin{aligned} \pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) &= \inf_N \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_n}^f} \inf_{|l| \leq N} E_Q[g - l\phi_{n+1}] \\ &= \lim_{N \rightarrow \infty} E_{Q_N}[g] - N|E_{Q_N}[\phi_{n+1}]| \\ &\leq \lim_{N \rightarrow \infty} E_{Q_N}[g]. \end{aligned}$$

The calibrating procedure described above yields $\lambda_N \in [0, 1]$ such that $\hat{Q}_N = (1 - \lambda_N)Q_N + \lambda_N \tilde{Q}_N \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f$. Moreover, as $|E_{Q_N}[\phi_{n+1}]| \rightarrow 0$ and \tilde{Q}_N can only be either Q_{\inf} or Q_{\sup} , these λ_N satisfy $\lambda_N \rightarrow 0$. This implies, $E_{\hat{Q}_N}[g] - E_{Q_N}[g] \rightarrow 0$ as $N \rightarrow \infty$ from which it follows

$$\pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) \leq \lim_{N \rightarrow \infty} E_{Q_N}[g] = \lim_{N \rightarrow \infty} E_{\hat{Q}_N}[g] \leq \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f} E_Q[g].$$

The converse inequality follows from standard arguments and hence we obtain $\pi_{\Omega_{\Phi}^*, \Phi_{n+1}}(g) = \sup_{Q \in \mathcal{M}_{\Omega_{\Phi}^*, \Phi_{n+1}}^f} E_Q[g]$ as required.

Finally we come to our initial assumption that $\Omega_{\Phi_n}^* \in \mathcal{F}^A$ for all $n \leq k$. Observe that in the setup of Proposition 2.6 this assumption is automatically satisfied since we show there that $\Omega = \Omega_{\Phi_n}^*$. Proposition 2.6 in turn allows us to establish Lemma 3.16 which, see Remark 3.17, then gives the required property $\Omega_{\Phi_n}^* \in \mathcal{F}^A$ for all $n \leq k$.

The proof is complete. ■

4.3 Proof of Theorem 2.9

We recall that the option ϕ_0 can be only bought at time $t = 0$ and the notations are as follows: $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^A) := \{(\alpha, H) \in \mathbb{R}_+ \times \mathcal{H}(\mathbb{F}^A)\}$ and $\widetilde{\mathcal{M}}_{\Omega, \phi_0} := \{Q \in \mathcal{M}_\Omega \mid E_Q[\phi_0] \leq 0\}$.

We first extend the results of Theorem 2.5 to the case where only ϕ_0 is available for static trading.

Lemma 4.2 *Suppose $\widetilde{\mathcal{M}}_{\Omega, \phi_0}^f \neq \emptyset$, $\pi_{\Omega^*}(\phi_0) > 0$ and $\Omega_{\phi_0}^* \in \mathcal{F}^A$. Then, for any \mathcal{F}^A -measurable g ,*

$$\pi_{\Omega_{\phi_0}^*, \phi_0}^*(g) = \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}^f} E_Q[g] = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g].$$

Proof. The idea of the proof is the same of Theorem 2.5. Suppose first that

$$\sup_{Q \in \mathcal{M}_\Omega^f} E_Q[\phi_0] > 0 \quad \text{and} \quad \inf_{Q \in \mathcal{M}_\Omega^f} E_Q[\phi_0] < 0. \quad (29)$$

Then it is easy to see that $\Omega_{\phi_0}^* = \Omega^*$. We use a variational argument to deduce the following equality:

$$\pi_{\Omega_{\phi_0}^*, \phi_0}^*(g) = \pi_{\Omega^*, \phi_0}(g) = \inf_N \sup_{Q \in \mathcal{M}_\Omega^f} (E_Q[g] - N|E_Q[\phi_0]|)$$

obtained with an application of min-max theorem (see Corollary 2 in Terkelsen (1972)). The last Step of the proof of Theorem 2.5 is only based on this variational equality and the analogous of (29). By repeating the same argument we obtain $\pi_{\Omega_{\phi_0}^*, \phi_0}(g) = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g]$. Since obviously

$$\pi_{\Omega_{\phi_0}^*, \phi_0}(g) = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g] \leq \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g] \leq \pi_{\Omega_{\phi_0}^*, \phi_0}(g)$$

we have the thesis.

Note now that the assumption $\pi_{\Omega^*}(\phi_0) > 0$ automatically implies $\sup_{Q \in \mathcal{M}_\Omega^f} E_Q[\phi_0] > 0$. Moreover, by assumption, $\widetilde{\mathcal{M}}_{\Omega, \phi_0}^f \neq \emptyset$ from which the inequality \leq for the infimum follows.

Suppose now $\inf_{Q \in \mathcal{M}_\Omega^f} E_Q[\phi_0] = 0$. From Proposition 4.1, $\pi_{\Omega^*}(-\phi_0) = 0$ and there exists a strategy $\bar{H} \in \mathcal{H}(\mathbb{F}^A)$ such that $(\bar{H} \circ S)_T \geq -\phi_0$ on Ω^* . We claim that the inequality is actually an equality on $\Omega_{\phi_0}^*$ (which is non-empty by assumption). If indeed for some $\omega \in \Omega_{\phi_0}^*$ the inequality is strict then, any $Q \in \mathcal{M}_\Omega^f$ such that $Q(\{\omega\}) > 0$, satisfies $E_Q[\phi_0] > 0$, which contradicts $\omega \in \Omega_{\phi_0}^*$. This implies that ϕ_0 is replicable on $\Omega_{\phi_0}^*$ and thus, $\widetilde{\mathcal{M}}_{\Omega, \phi_0}^f = \mathcal{M}_{\Omega, \phi_0}^f = \mathcal{M}_{\Omega_{\phi_0}^*}^f$. In such a case,

$$\pi_{\Omega_{\phi_0}^*, \phi_0}(g) \leq \pi_{\Omega_{\phi_0}^*}^*(g) = \sup_{Q \in \mathcal{M}_{\Omega_{\phi_0}^*}^f} E_Q[g] = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g],$$

where, the first equality follows from Proposition 4.1, since $\Omega_{\phi_0}^* \in \mathcal{F}^A$ by assumption. The thesis now follows from standard arguments as above.

■

Proposition 4.3 *Assume that Ω satisfies that there exists an ω^* such that $S_0(\omega^*) = S_1(\omega^*) = \dots = S_T(\omega^*)$, $\Omega = \Omega^*$ and $\pi_{\Omega^*}(\phi_0) > 0$. Then the following are equivalent:*

- (1) *There is no Uniformly Strong Arbitrage on Ω in $\widetilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^A)$;*

(2) There is no Strong Arbitrage on Ω in $\tilde{\mathcal{A}}_{\phi_0}(\mathbb{F}^A)$;

(3) $\tilde{\mathcal{M}}_{\Omega, \phi_0} \neq \emptyset$.

Moreover, when any of these holds, for any upper semi-continuous $g : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}$ such that

$$\lim_{|x| \rightarrow \infty} \frac{g^+(x)}{m(x)} = 0, \quad (30)$$

where $m(x_0, \dots, x_T) := \sum_{t=0}^T g_0(x_t)$, we have the following pricing-hedging duality:

$$\pi_{\Omega^*, \phi_0}(g(S)) = \sup_{Q \in \tilde{\mathcal{M}}_{\Omega, \phi_0}} E_Q[g(S)] = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S)]. \quad (31)$$

Remark 4.4 We observe that the assumption $\pi_{\Omega^*}(\phi_0) > 0$ is not binding and can be removed. In fact if $\pi_{\Omega^*}(\phi_0) \leq 0$, (1) \Rightarrow (3) is obviously satisfied since $\tilde{\mathcal{M}}_{\Omega, \phi_0} = \mathcal{M}_{\Omega} \neq \emptyset$. The difference is that the pricing-hedging duality (31) is (trivially) satisfied only in the first equation.

Proof of Proposition 4.3.

(3) \Rightarrow (2) and (2) \Rightarrow (1) are obvious.

To show (1) \Rightarrow (3), we suppose there is no Uniformly Strong Arbitrage on Ω in $\tilde{\mathcal{A}}_{\phi_0}(\tilde{\mathbb{F}})$.

We first show that the interesting case is $\pi_{\Omega^*}(\phi_0) > 0$ and $\pi_{\Omega^*}(-\phi_0) = 0$. The other cases follows trivially from Proposition 4.1 and Lemma 4.2:

- If $\pi_{\Omega^*}(-\phi_0) < 0$, since the superhedging price is attained and $\Omega = \Omega^*$, there exist $H \in \mathbb{F}^A$ and $x < 0$ such that

$$\phi_0(\omega) + (H \circ S)_T(\omega) \geq -x > 0, \quad \forall \omega \in \Omega$$

which is clearly a Uniform Strong Arbitrage on Ω .

- If $\pi_{\Omega^*}(-\phi_0) > 0$ and $\pi_{\Omega^*}(\phi_0) > 0$, we have that 0 is in the interior of the price interval formed by $\inf_{Q \in \mathcal{M}_{\Omega}^f} E_Q[\phi_0]$ and $\sup_{Q \in \mathcal{M}_{\Omega}^f} E_Q[\phi_0]$. Thus, $\tilde{\mathcal{M}}_{\Omega, \phi_0}^f \supseteq \mathcal{M}_{\Omega, \phi_0}^f \neq \emptyset$ and, it is straightforward to see that $\Omega_{\phi_0}^* = \Omega^*$.

Note that in all these cases $\Omega_{\phi_0}^* = \Omega^* \in \mathcal{F}^A$ and hence (31) follows from Lemma 4.2.

The remaining case is $\pi_{\Omega^*}(\phi_0) > 0$ and $\pi_{\Omega^*}(-\phi_0) = 0$. In this case, by considering the ω^* such that $s_0 = S_0(\omega^*) = S_1(\omega^*) = \dots = S_T(\omega^*)$, we observe that the super-replication of $-\phi_0$ necessarily requires an initial capital of, at least, $-g_0(s_0)$. From $\pi_{\Omega^*}(-\phi_0) = 0$ we can rule out the possibility that $g_0(s_0) < 0$. Note now that, by the convexity of g_0 , for any $l \in 0, \dots, T-1$, $g_0(S_T(\omega)) - \sum_{i=l+1}^T g'_0(S_{i-1}(\omega))(S_i(\omega) - S_{i-1}(\omega)) \geq g_0(S_l(\omega))$ for any $\omega \in \Omega$. In particular, when $l = 0$,

$$g_0(S_T(\omega)) - \sum_{i=1}^T g'_0(S_{i-1}(\omega))(S_i(\omega) - S_{i-1}(\omega)) \geq g_0(s_0) \quad \forall \omega \in \Omega. \quad (32)$$

Denote by \bar{H} the dynamic strategy in (32). If $g_0(s_0) > 0$, $(1, \bar{H})$ is a Uniformly Strong Arbitrage on Ω and hence, a contradiction to our assumption. Thus, $g_0(s_0) = 0$. In this case, it is obvious that the Dirac measure $\delta_{\omega^*} \in \mathcal{M}_{\Omega, \phi_0}^f \subseteq \tilde{\mathcal{M}}_{\Omega, \phi_0}^f$ which is therefore non-empty.

Moreover, since $\delta_{\omega_*} \in \mathcal{M}_{\Omega, \phi_0}^f$,

$$\sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}^f} E_Q[g(S)] \geq \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g(S)] \geq g(s_0, \dots, s_0).$$

Case 1. Suppose g is bounded from above. We show that it is possible to super-replicate g with any initial capital larger than $g(s_0, \dots, s_0)$. To see this recall that, from the strict convexity of g_0 , the inequality in (32) is strict for any $s \in \mathbb{R}_+^{d \times (T+1)}$ such that s is not a constant path, i.e., $s_i \neq s_0$ for some $i \in \{0, \dots, T\}$. In fact, it is bounded away from 0 outside any small ball of (s_0, \dots, s_0) . Hence, due to the upper semi-continuity and boundedness of g , for any $\varepsilon > 0$, there exists a sufficiently large K such that

$$g(s_0, \dots, s_0) + \varepsilon + K \left\{ g_0(S_T(\omega)) - \sum_{i=1}^T g'_0(S_{i-1}(\omega))(S_i(\omega) - S_{i-1}(\omega)) \right\} \geq g(S(\omega)) \quad \forall \omega \in \Omega^*$$

Therefore, $\pi_{\Omega^*, \phi_0}(g) \leq g(s_0, \dots, s_0) \leq \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g(S)] \leq \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}^f} E_Q[g(S)]$. The converse inequality is easy and hence we obtain $\pi_{\Omega^*, \phi_0}(g) = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g(S)] = \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}^f} E_Q[g(S)]$ as required.

Case 2. It remains to argue that the duality still holds true for any g that is upper semi-continuous and satisfies (30). We first argue that any upper semi-continuous $g : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}$, satisfying (30), can be super-replicated on Ω^* by a strategy involving dynamic trading in S , static hedging in g_0 and cash. Define a synthetic option with payoff $\tilde{m} : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}$ by

$$\tilde{m}(x_0, \dots, x_T) = \sum_{l=0}^T \left\{ g_0(x_T) - \sum_{i=l+1}^T g'_0(x_{i-1})(x_i - x_{i-1}) \right\}. \quad (33)$$

By convexity of g , we know that

$$\tilde{m}(x_0, \dots, x_T) = \sum_{l=0}^T \left\{ g_0(x_T) - \sum_{i=l+1}^T g'_0(x_{i-1})(x_i - x_{i-1}) \right\} \geq \sum_{l=0}^T g_0(x_l) = m(x_0, \dots, x_T).$$

Since we assume there is no Uniform Strong Arbitrage, it is clear that $\pi_{\Omega^*, \phi_0}(\tilde{m}(S)) = 0$.

From (30) it follows that $g(S) - \tilde{m}(S)$ is bounded from above. By sublinearity of $\pi_{\Omega^*, \phi_0}(\cdot)$, we have

$$\begin{aligned} \pi_{\Omega^*, \phi_0}(g) &\leq \pi_{\Omega^*, \phi_0}(g(S) - \tilde{m}(S)) + \pi_{\Omega^*, \phi_0}(\tilde{m}(S)) \\ &= \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S) - \tilde{m}(S)] + 0 \\ &= \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S)] \leq \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}} E_Q[g(S)], \end{aligned}$$

where the first equality follows from the pricing-hedging duality for claims bounded from above, that we established in Case 1, and the fact that $\pi_{\Omega^*, \phi_0}(\tilde{m}(S)) = 0$. Moreover, for $Q \in \mathcal{M}_{\Omega, \phi_0}$, $E_Q[\tilde{m}(S)] = 0$ from which the second equality follows.

The converse inequality follows from standard arguments and hence we have obtained $\pi_{\Omega^*, \phi_0}(g) = \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}^f} E_Q[g(S)] = \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}^f} E_Q[g(S)]$ as required. \blacksquare

Proof of Theorem 2.9.

(3) \Rightarrow (2) and (2) \Rightarrow (1) are obvious.

Step 1. To show that (1) implies (3). Suppose there is no Uniformly Strong Arbitrage on Ω in $\mathcal{A}_\Phi(\tilde{\mathbb{F}})$. We first know from Proposition 4.3 that $\mathcal{M}_{\Omega, \phi_0} \neq \emptyset$. We can use a variational argument to deduce the following equalities: fix an arbitrary $K > 0$ and let $\tilde{m} : \mathbb{R}_+^{d \times (T+1)} \rightarrow \mathbb{R}$ defined as in (33)

$$\begin{aligned} \pi_{\Omega^*, \Phi}(g(S)) &= \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} \pi_{\Omega^*, \phi_0}(g(S) - X) \\ &= \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S) - X] \end{aligned} \quad (34)$$

$$= \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S) - X - K\tilde{m}(S)] \quad (35)$$

where the second equality follows from Proposition 4.3. Denote by \mathcal{Q} (respectively $\tilde{\mathcal{Q}}$) the set of law of S under the measures $Q \in \mathcal{M}_{\Omega, \phi_0}$ (respectively $Q \in \tilde{\mathcal{M}}_{\Omega, \phi_0}$) and write $\text{Lin}(\{g_i\}_{i \in I/\{0\}})$ for the set of finite linear combinations of elements in $\{g_i\}_{i \in I/\{0\}}$. Observe that from Step 1 of the proof of Theorem 1.3 in Acciaio et al. (2016) $\tilde{\mathcal{Q}}$ is weakly compact and hence the same is true for $\overline{\mathcal{Q}}$ (the weak closure of \mathcal{Q}).

By a change of variable we have

$$\inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S) - X - K\tilde{m}(S)] = \inf_{G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})} \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\tilde{g}(\mathbb{S})],$$

where $\mathbb{S} := (\mathbb{S}_t)_{t=0}^T$ is the canonical process on $\mathbb{R}_+^{d \times (T+1)}$ and $\tilde{g} = g - G - K\tilde{m}$. We aim at applying min-max theorem (see Corollary 2 in Terkelsen (1972)) to the compact convex set $\overline{\mathcal{Q}}$, the convex set $\text{Lin}(\{g_i\}_{i \in I/\{0\}})$, and the function

$$f(\mathbb{Q}, G) = \int_{\mathbb{R}_+^{d \times (T+1)}} \left(g(s_0, \dots, s_T) - G(s_0, \dots, s_T) - K\tilde{m}(s_0, \dots, s_T) \right) d\mathbb{Q}(s_0, \dots, s_n).$$

Clearly f is affine in each of the variables. Furthermore, we show that $f(\cdot, G)$ is upper semi-continuous on $\overline{\mathcal{Q}}$. To see this, fix $G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})$. By definition of f we have that

$$f(\mathbb{Q}, G) = \mathbb{E}_{\mathbb{Q}}[\tilde{g}(\mathbb{S})]. \quad (36)$$

It follows from Assumption 2.8 and (8) that \tilde{g} is bounded from above. Hence, for every sequence of $\{\mathbb{Q}_n\}_n \in \overline{\mathcal{Q}}$ with $\mathbb{Q}_n \rightarrow \mathbb{Q}$ as $n \rightarrow \infty$ for some \mathbb{Q} weakly, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_n}[\tilde{g}^+(\mathbb{S})] \leq \mathbb{E}_{\mathbb{Q}}[\tilde{g}^+(\mathbb{S})]$$

by Portmanteau theorem, and

$$\liminf_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_n}[\tilde{g}^-(\mathbb{S})] \geq \mathbb{E}_{\mathbb{Q}}[\tilde{g}^-(\mathbb{S})]$$

by Fatou's lemma, where $\tilde{g} := \tilde{g}^+ - \tilde{g}^-$ with $\tilde{g}^+ := \max\{\tilde{g}, 0\}$, $\tilde{g}^- := (-\tilde{g})^+$. Then

$$\limsup_{n \rightarrow \infty} f(\mathbb{Q}_n, G) = \limsup_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_n}[\tilde{g}(\mathbb{S})] \leq \mathbb{E}_{\mathbb{Q}}[\tilde{g}(\mathbb{S})] = f(\mathbb{Q}, G).$$

Therefore, the assumptions of Corollary 2 in Terkelsen (1972) are satisfied and we have, by recalling $\mathcal{Q} \subseteq \overline{\mathcal{Q}}$ and equation (35),

$$\begin{aligned}
\pi_{\Omega^*, \Phi}(g(S)) &= \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} \sup_{Q \in \mathcal{M}_{\Omega, \phi_0}} E_Q[g(S) - X - K\tilde{m}(S)] \\
&= \inf_{G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[g(\mathbb{S}) - G(\mathbb{S}) - K\tilde{m}(\mathbb{S})] \\
&\leq \inf_{G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})} \sup_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[g(\mathbb{S}) - G(\mathbb{S}) - K\tilde{m}(\mathbb{S})] \\
&= \sup_{Q \in \overline{\mathcal{Q}}} \inf_{G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})} \mathbb{E}_Q[g(\mathbb{S}) - G(\mathbb{S}) - K\tilde{m}(\mathbb{S})] \\
&\leq \sup_{Q \in \overline{\mathcal{Q}}} \inf_{G \in \text{Lin}(\{g_i\}_{i \in I/\{0\}})} \mathbb{E}_Q[g(\mathbb{S}) - G(\mathbb{S}) - K\tilde{m}(\mathbb{S})] \\
&= \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}} \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} E_Q[g(S) - X - K\tilde{m}(S)]. \tag{37}
\end{aligned}$$

Take $g = 0$. If $\widetilde{\mathcal{M}}_{\Omega, \Phi} = \emptyset$, then

$$\sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \phi_0}} \inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} E_Q[-X - K\tilde{m}(S)] = -\infty,$$

and hence $\pi_{\Omega^*, \Phi}(0) = -\infty$, which contradicts the no arbitrage assumption. Therefore we have (1) implies (3).

Step 2. To show the pricing–hedging duality, suppose now $\widetilde{\mathcal{M}}_{\Omega, \Phi} \neq \emptyset$. If $Q \notin \widetilde{\mathcal{M}}_{\Omega, \Phi}$, then

$$\inf_{X \in \text{Lin}(\Phi/\{\phi_0\})} E_Q[g(S) - X - K\tilde{m}(S)] = -\infty.$$

Therefore, in (37), it suffices to look at measures in $\widetilde{\mathcal{M}}_{\Omega, \Phi} \neq \emptyset$ only, and hence we obtain

$$\begin{aligned}
\pi_{\Omega^*, \Phi}(g(S)) &\leq \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[g(S) - K\tilde{m}(S)] \\
&\leq \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[g(S)] + K \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[-\tilde{m}(S)] \\
&= \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[g(S)] + K(T+1) \sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[-g_0(S)]
\end{aligned}$$

Since $-g_0$ is bounded from above, the quantity $\sup_{Q \in \widetilde{\mathcal{M}}_{\Omega, \Phi}} E_Q[-g_0(S)]$ is finite and, by recalling that $K > 0$ is arbitrary, we get the thesis for $K \downarrow 0$. ■

5 Appendix

Lemma 5.1 *Let $f : X \mapsto \mathbb{R}^n$ be a \mathcal{G} -measurable function with \mathcal{G} a sub sigma-algebra of \mathcal{B}_X . Let $(\mathcal{G})^{\mathcal{A}}$ (resp. $(\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}}$) the sigma-algebra generated by the analytic, \mathcal{G} -measurable (resp. $\mathcal{B}_{\mathbb{R}^n}$ -measurable), subsets of X (resp. \mathbb{R}^n). Then*

1. *for every $A \in (\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}}$ we have $f^{-1}(A) \in (\mathcal{G})^{\mathcal{A}}$;*
2. *for every $A \in (\mathcal{G})^{\mathcal{A}}$ we have $f(A) \in (\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}}$.*

Proof. From Theorem III.11 and Theorem III.18 in Dellacherie and Meyer (1982), both claims hold for the analytic sets which are the generators of the respective sigma-algebra. We therefore have that the sets

$$\begin{aligned}\{A \in (\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}} \mid f^{-1}(A) \in (\mathcal{G})^{\mathcal{A}}\} &\subseteq (\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}} \\ \{A \in (\mathcal{G})^{\mathcal{A}} \mid f(A) \in (\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}}\} &\subseteq (\mathcal{G})^{\mathcal{A}}\end{aligned}$$

contain the analytic subsets of \mathbb{R}^n and X , respectively. Since they are both sigma-algebras they also contain $(\mathcal{B}_{\mathbb{R}^n})^{\mathcal{A}}$ and $(\mathcal{G})^{\mathcal{A}}$ respectively, from which the thesis follows. ■

5.1 Remark on conditional supports

Let $\mathcal{G} \subseteq \mathcal{B}_X$ be a countably generated sub σ -algebra of \mathcal{B}_X . Then there exists a proper regular conditional probability, i.e. a function $\mathbb{P}_{\mathcal{G}}(\cdot, \cdot) : (X, \mathcal{B}_X) \mapsto [0, 1]$ such that:

- a) for all $\omega \in \Omega$, $\mathbb{P}_{\mathcal{G}}(\omega, \cdot)$ is a probability measure on \mathcal{B}_X ;
- b) for each (fixed) $B \in \mathcal{B}_X$, the function $\mathbb{P}_{\mathcal{G}}(\cdot, B)$ is \mathcal{G} -measurable and a version of $E_{\mathbb{P}}[\mathbf{1}_B \mid \mathcal{G}](\cdot)$ (here the null set where they differ depends on B);
- c) $\exists N \in \mathcal{G}$ with $\mathbb{P}(N) = 0$ such that $\mathbb{P}_{\mathcal{G}}(\omega, B) = \mathbf{1}_B(\omega)$ for $\omega \in X \setminus N$ and $B \in \mathcal{G}$ (here the null set where they differ does not depend on B); moreover for all $\omega \in X \setminus N$, we have $\mathbb{P}_{\mathcal{G}}(\omega, A_{\omega}) = 1$ where $A_{\omega} = \bigcap \{A : \omega \in A, A \in \mathcal{G}\} \in \mathcal{G}$.
- d) For every $A \in \mathcal{G}$ and $B \in \mathcal{B}_X$ we have $\mathbb{P}(A \cap B) = \int_A \mathbb{P}_{\mathcal{G}}(\omega, B) \mathbb{P}(d\omega)$

We now consider a measurable $\xi : X \rightarrow \mathbb{R}^d$ and $P_{\xi} : X \times \mathcal{B}_{\mathbb{R}^d} \rightarrow [0, 1]$ defined by

$$P_{\xi}(\omega, B) := \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_{\omega} \mid \xi(\tilde{\omega}) \in B\}),$$

and observe that from a) and with N as in c), for any $\omega \in X \setminus N$, $P_{\xi}(\omega, \cdot)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. Finally, we let $B_{\varepsilon}(x)$ denote the ball of radius ε with center in x , and we introduce the closed valued random set

$$\omega \mapsto \chi_{\mathcal{G}}(\omega) := \{x \in \mathbb{R}^d \mid P_{\xi}(\omega, B_{\varepsilon}(x)) > 0 \forall \varepsilon > 0\}, \quad (38)$$

for $\omega \in X \setminus N$ and \mathbb{R}^d otherwise. $\chi_{\mathcal{G}}$ is \mathcal{G} -measurable since, for any open set $O \subseteq \mathbb{R}^d$, we have

$$\{\omega \in X \mid \chi_{\mathcal{G}}(\omega) \cap O \neq \emptyset\} = N \cup \{\omega \in X \setminus N \mid P_{\xi}(\omega, O) > 0\} = N \cup \{\omega \in X \setminus N \mid \mathbb{P}_{\mathcal{G}}(\omega, \xi^{-1}(O) \cap A_{\omega}) > 0\},$$

with the latter belonging to \mathcal{G} from b) and c) above. By definition $\chi_{\mathcal{G}}(\omega)$ is the support of $P_{\xi}(\omega, \cdot)$ and therefore, for every $\omega \in X$, $P_{\xi}(\omega, \chi_{\mathcal{G}}(\omega)) = 1$. Notice that since the map $\chi_{\mathcal{G}}$ is \mathcal{G} -measurable then for $\omega \in X$ we have $\chi_{\mathcal{G}}(\omega) = \chi_{\mathcal{G}}(\tilde{\omega})$ for all $\tilde{\omega} \in A_{\omega}$.

Lemma 5.2 *Under the previous assumption we have $\{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\} \in \mathcal{B}_X$ and $\mathbb{P}(\{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\}) = 1$.*

Proof. Set $B := \{\omega \in X \mid \xi(\omega) \in \chi_{\mathcal{G}}(\omega)\}$. $B \in \mathcal{B}_X$ follows from the measurability of ξ and $\chi_{\mathcal{G}}$. From the properties of regular conditional probability we have

$$\mathbb{P}(B) = \mathbb{P}(B \cap X) = \int_X \mathbb{P}_{\mathcal{G}}(\omega, B) \mathbb{P}(d\omega).$$

Consider the atom $A_{\omega} = \cap\{A : \omega \in A, A \in \mathcal{G}\}$. From property c) we have $\mathbb{P}_{\mathcal{G}}(\omega, A_{\omega}) = 1$ for any $\omega \in X \setminus N$. Therefore for every $\omega \in X \setminus N$ we deduce

$$\begin{aligned} \mathbb{P}_{\mathcal{G}}(\omega, B) &= \mathbb{P}_{\mathcal{G}}(\omega, B \cap A_{\omega}) = \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_{\omega} \mid \xi(\tilde{\omega}) \in \chi_{\mathcal{G}}(\tilde{\omega})\}) \\ &= \mathbb{P}_{\mathcal{G}}(\omega, \{\tilde{\omega} \in A_{\omega} \mid \xi(\tilde{\omega}) \in \chi_{\mathcal{G}}(\omega)\}) = P_{\xi}(\omega, \chi_{\mathcal{G}}(\omega)) = 1. \end{aligned}$$

Therefore

$$\mathbb{P}(B) = \int_X \mathbb{P}_{\mathcal{G}}(\omega, B) \mathbb{P}(d\omega) = \int_X \mathbf{1}_{X \setminus N}(\omega) \mathbb{P}(d\omega) = 1.$$

■

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